Archives for the Philosophy and History of Soft Computing

European Centre for Soft Computing
www.softcomputing.es

Issue 1/2015

INTERNATIONAL ONLINE JOURNAL
OPEN ACCESS
ISSN 2341-0183

Fuzzy-set representation and processing of fuzzy images: non-linguistic vagueness as representation, approximation and scientific practice (Part 1)

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Abstract This is the first part of a two-part paper in which I conclude the process, initiated elsewhere, of tracking objective conditions of vagueness of representation from language to pictures, from philosophy to imaging science, from vagueness to approximation, from representation to reasoning, with a focus on the application of fuzzy set theory and its challenges.

Submitted: 01/07/15. Accepted and Published: 31/07/15

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1 Introduction: from philosophy of language, science and art to mathematics

In this two-part paper I conclude the process initiated elsewhere[1] of tracking objective conditions of vagueness of representation from language to pictures, from philosophy to imaging science, from vagueness to approximation, from representation to reasoning, with a focus on the application of fuzzy set theory and its challenges. Reference to a previous discussion elsewhere will be necessary to make my arguments below appear motivated, informative and interesting.[2]

Fuzziness has been the central formal notion of indeterminacy of linguistic and pictorial representation and of its interpretation in terms of objective categorizations or inherent properties. Next I argue that the application of mathematics to the previous conclusions about pictorial vagueness yields interesting lessons about fuzziness in pictures and the application of fuzzy set theory, in particular, and of mathematics, more generally. I note that while fuzzy modeling offers a unified framework and precedent for treating vagueness in words and images, unity of application isn't simple. Despite a shared interpretive approach based on a formal modeling of categorization, the application of fuzzy sets in the case of images presents distinctive features and challenges. The previous results, I suggest, yield lessons for interpreting fuzzy theory and applying it, especially in pictorial representation and reasoning.

Mathematics is, as a practice, application through and through; even in the practice of the pure mathematics and language isolated in late-19th-century formalism and early-20th-century linguistic turns.[3] But, as I have indicated, also the application of fuzzy formalism to pictures precedes my introduction here in philosophical discussion; in particular, it can be found in the analysis and processing of images. The practice of processing images, especially fuzzy ones, is a material extension of formal mathematical practices, offering an empirical extension of its technological objectivity.

(A)-Summary of previous questions and results.[4]

A1- A longstanding issue in the philosophy of language is the characterization and interpretation of vagueness –other philosophical and practical issues involving reasoning, decision-making, etc. depend on views

[1] Cat forthcoming, Part 1.

[2] Alternatively, my discussion might be easily mistaken for a summary presentation of elementary technical notions and results.

[3] Cat 2015a and 2015b.

[4] Cat forthcoming.

on vagueness. The issue is framed in semantic terms of meaning-fixing practices; and it has long and typically been settled on the metaphysical grounds: that things have determinate properties corresponding to precise concepts, with the subjectivist, anti-realist conclusion that vagueness is an epistemic condition of the speaker. Beyond this, the debate is dominated by discussions of the relevant sort of subjectivism with defenses and criticisms of particular positions.

A recent discussion has distinguished two problems associated with vagueness, the location problem –locating boundaries of predicates– and the jolt problem –determining the sharpness of the boundary.[5] We find available also four criteria of vagueness: borderline cases, sorites paradoxes, blurred extension boundaries and closeness in respect of truth. Defending closeness of truth as a general criterion addresses the so-called jolt problem, the problem of indeterminacy (rather than the epistemic problem of knowing where the boundaries lie), and allows for a defense of an objective semantic interpretation of vagueness. The strategy of interpreting vague predicates as representations of properties of systems is based on the way associating predicates with fuzzy sets provides a criterion of truth degrees.[6]

A2-Based on previous work,[7] I adopt fuzzy set theory as a model of practices and processes of categorization (not just a formal representation of categories); this move allows me to defend a polar, dual subjective-objective view of vagueness as fuzziness, as an alternative to the dichotomy between objective and subjective interpretations (replacing a series of associated dichotomies, subjective/objective, predicate/property, cognitive/ontic, etc , within a framework of empirical, material, social objectivity of categorization practices).

A3-Are criteria of pictorial vagueness possible? I suggest a strategy based on a focus on categorization and the extended application of fuzziness. And I point to ways this move has conceptual implications for the philosophical project to relate vagueness and imagery, also to understand the application of fuzzy set theory. It also has conceptual, methodological and practical lessons for the application.

A4-Pictures differ from predicates. Representation, or depiction, is not an exclusive function of images. Still, depiction alone isn't a simple matter. An extensive literature from the last five decades, especially in philosophy, cognitive sciences, cultural studies and art history, suggests that depiction depends on a complex of conditions and activities involving

[5] Smith 2010.
[6] Ibid.
[7] Cat 2015a and 2015b.

authors and viewers alike, including material media, techniques, conventions, institutions, perceptual and learning processes, intentionality, etc.

From a semantic standpoint, depiction escapes the accounts of how predicates describe, names refer or propositions are truth-bearers. Pictorial representation is not truth. Pictorial depiction is not reducible to the representational functionality of symbolic systems. An image, or icon, is not truth-functional; its semantic function and success mobilize alternative or additional semantic or epistemic kinds of contents or relations.

I admit of varieties of modes of depiction to do justice to different representational and referential practices and standards, also their combined use (I take a broad scope of combined uses to include the juxtaposition of the pictorial and the symbolic, as in the role of titles and captions). Possible conditions include a role, *inter alia*, for convention (in the manner of symbols), indexicality (causal reference in Peirce's sense of indexical signs, linked to their subject through acknowledged causal or regular link), analogy (role of resemblance in icons), make-believe (Walton), seeing-in and seeing-as (Wollheim), instantiation (Goodman) and recognitional capacities (Lopes).[8] In pictorial representation and reference, perceptual depiction admits of formal modeling by fuzzy set theory, but only in connection with types of accounts that emphasizes a role for perceptual categorization, such as instantiation, seeing-in/as and recognitional skills. This role for categorization bridges the gap between predicates and pictures.

Holism and atomism set limits to the validity of the so-called Fodor-Sober criterion of pictorial representation (in the analogical continuum). The Fodor-Sober criterion distinguishes images from symbolic forms of representation in spatial mereological terms: for an image to represent a system, its spatial parts must represent equally the corresponding extended parts of the system.[9] Holism in perceptual recognition adds complexity to this constraint on depiction and discrimination, for instance through environmental contexts, neural or Gestalt-like patterning. Also molecular structures of the medium, from natural granularity in photographic plate or paper to digital pixelation on electronic screens, set a lower limit to discrimination or resolution.

A5-Visual cognition: from images to pictures, practices and processes with cognitive functions involve visual representations, whether in internal images of perception or in external images on material media, or

[8] See for instance Goodman 1976, Wollheim 1980, Walton 1990, Lopes 1996 and Kulvicki 2014.
[9] See Sober 1976.

pictures.[10] In a broad variety of contexts and activities we seek and rely on visual perception and visual representation for the sake of communication and visual thinking. As examples of thinking I include activities of problem-solving and inference (computation might include both). Note that the success of producing, understanding and otherwise using representations might itself be the outcome of reasoning and problem-solving (and vice versa).

A6-Pictorial empiricism in science: at the visual level, also the production, defense and communication of scientific ideas and scientific evidence extend beyond the recourse to direct observations of visible properties of systems or phenomena of interest or, in general, to aided activities of perceptual inquiry. We may speak of visual data in a in immediate perceptual sense, when visible properties are of interest, and more technologically-based, when circulated records that are the outcome of detection or measurement techniques for the sake of exploration and evidence have visual character.[11] Whenever the relevant data has symbolic or any non-visual character but are given visual representation, we speak of visualized data.

Not only data have visual character. A great deal of scientific pictures are illustrations of symbolic or conceptual relations.[12] If speak of models as limited-use idealized conceptions or hypotheses about systems or phenomena of interest, we encounter visual models. Given the use of graphic representations of data, models and diagrams for a variety of cognitive aims such as computation, explanation, prediction, we may speak of scientific visual cognition, one based on pictures and not just perceptual images. Pictorial empiricism relies on practices with the complexity of relevant skills, techniques, media and values. The concern with prescriptive constraints guiding the production of visual graphic representations suggests the normative dimension.[13]

A7-Pictures have content. In other words, considerations of precision require considerations of content, which introduces a restriction on my analysis of pictures. Content is often implied by evaluations of accuracy in

[10] Only the computational priority of symbolic processing has been the subject of heated scientific controversy since the 1980s; see Kosslyn 1994 and Rollins 1989.

[11] For a complex account of data noting their material, theoretical and social grounds, see Leonelli 2012 and forthcoming.

[12] In this sense, when underlying thoughts involved in the process of producing empirical data, visualized data may qualify as well as just symbolic relations such as algebraic equations or formal rules or more descriptive conceptions of empirical or theoretical phenomena.

[13] For instance, see Tufte's guiding rules to avoid lying with graphs in Tufte 1991.

pictures, including accuracy of perceptions.[14] The identification of content is often associated with activities of categorization and the conditions of exercise of recognitional skills. Then the role of categorization in establishing content is what allows for considerations of precision.

I distinguish between intrinsic and extrinsic content. Intrinsic content is established in relation to visual categorization, the identification of what in a given context operate as elementary visual features. Any additional content is extrinsic content, established through transparent or opaque processes or associations (inferential, conceptual or otherwise). In given limited contexts, stable or standard links for content development may be adopted; I call them intrinsic-extrinsic content links, or IC-EC links for short (formalized versions may be called IC-EC rules). The adopted links may be supported by empirical regularities, theoretical hypotheses, techniques, etc. with restricted validity or relevance.

A9-Vague or fuzzy pictures have vague or fuzzy content. From a perceptual standpoint, the characteristic we call blur is a form of imprecise intrinsic content and categorical vagueness (both pictorial and perceptual). This is related to a fourfold phenomenological equivalence between different categorization situations: seeing something fuzzy (relative to a descriptive material hypothesis), seeing fuzzy something sharp (also relative to a material hypothesis), seeing a fuzzy picture and seeing fuzzy a sharp picture.

A10-Form of pictorial vagueness: the extended, non-linguistic application of fuzzy-set theory suggests an exclusive attention to categorization-based kinds of depiction (besides description) –hence the attention restricted to content. Yet pictorial depiction is not truth-functional, its semantic contents is based o another semantic or epistemic kind of relation. This suggests a reformulation of two categorization-based criteria of vagueness, borderline and closeness. If pictorial vagueness is pictorial fuzziness, pictorial closeness cannot be reduced to closeness in truth, expressed in terms of truth degrees (of propositions involving fuzzy-set predicates).

A11-Conceptual models and technological practices of fuzziness –such as analysis and processing– rely on an emphasis on two kinds of IC-EC links: PIC-PEC and IIC-IEC links or rules. PIC-PEC rules associate systematically precise extrinsic content with precise intrinsic content; IIC-IEC rules associate imprecise extrinsic content with imprecise intrinsic content. Additional semantic pairing (meta)rules are: precision-accuracy and imprecision-inaccuracy.

[14] A general defense of this view can be found in Siegel 2010; I consider it only in the context of the applicability of fuzzy sets as a model of visual vagueness or fuzziness in pictures.

A12-Challenges to PEC-PEC and IIC-IEC rules. Alternative rules such as of the PEC-IIC form operate in representational practices and processes, as adopted in the historical record on practices and technologies of graphic depiction and in modern cognitive psychology. Their application is supported by commitments to material and perceptual modes of realism: fuzzy images have been long endorsed as a precise representation of how we perceive motion, distance or the focus of attention, or else how, from a material perspective, how fog, steam or furry textures are precisely describable on a given scale. In addition, each kind of commitment illustrates a corresponding kind of objectivity of vagueness, that is, of vague categories or features. Accuracy of perceptual or material representation often ensues. Fuzziness enforces exactness. We can add, then, two more semantic rules: precision-inaccuracy and imprecision-accuracy. More generally, the cognitive, social and aesthetic meanings of fuzziness in practice are diverse.[15]

(B)-In image processing the constructed visual representation plays an instrumental role as blueprint for control of the image, in particular of the intrinsic content for the sake of fixing the extrinsic one. After assuming what I have called an IC-EC link, according to which intrinsic features contribute to the recognition of extrinsic ones, the image content, the reduction of vagueness –even inaccuracy– in one is taken to serve the purpose of reducing it in the other. Also in the fuzzy case I want to challenge the systematic application of the particular form of the condition I have called the PIC-PEC rule –and the imprecision-inaccuracy metarule.

If I am right, the discussion above leads to four main conclusions that speak to the specificity of pictures and the richness of mathematical treatments.

(B1)-In the idealized case of imagery as a well-formed language, the process of formal application is directed towards the possibility of effective control; moreover, the aim and practice of control is inseparable from techniques and standards of calculation and representation. That is, fixing pictorial contents is part of a richer practice involving formal and technological processing, and this practice takes place within contexts informed by aims and standards of categorization, understanding and practical application, along with key roles for rules, techniques, hypotheses,

[15] On the cognitive science perspective on graphic design see Ware 2008; for a sense of diversity and debate in painting and photography see Eastlake 1857, Emerson 1889, Hamilton 2001 Hentschel 2001 and Richter 2009. Blurring has also additional uses with social meaning such as masking for the sake of protection or censure.

judgments and decisions.[16]

We may equally identify prescriptive and pragmatic dimensions of uses of formalism beyond their empirical applications, engrained more deeply already in formal practices (including graphic representation). Here, we should distinguish between formal and symbolic aspects of mathematical practices of representation, reasoning and computation. Formalism includes alongside a system of symbols a set of explicit rules for constructing well-formed expressions, some of which function as rules of proof or inference. This is significant because it is the formal dimension that enforces the prescriptive role of mathematical structures, including the ideal of axiomatics.[17]

Part and parcel of process of formalizing the cognitive role of vagueness is the application of formal methods to fixing, coordinating and extending content through formal models of categorization. Different formal frameworks are available, and for the sake of modeling and controlling vagueness some involve generalizations of set theory, including, but not only, fuzzy sets. Processes of cognition and technological processing may be formalized in such a way by considering rules and techniques of analysis and synthesis; I argue they are not reducible to mere compositionality. Some such rules and procedures connect what I call intrinsic and extrinsic content. In addition, I argue that they satisfy different criteria of objectivity.

(B2)-One distinguishing feature of the application of fuzzy sets to modeling vague pictorial representation stems from the role of the duality of types of contents in making representation possible and the particular role of categorization in fixing contents. Within the common framework of fuzzy categorization, the application of the linguistic standard is hardly uniform, that is, uniformly direct. I argue that the relation of fuzzy pictorial representation is not generally reducible to a simple semantic relation of

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[16] On such roles in mathematical application see Cat 2015a and 2015b. In connection with the methodological value of ambiguity, Grosholz has defended the notion of productive ambiguity: A systematic use of ambiguity scientific consists in the joint use of a multiplicity of modes of representation and argumentation in problem-solving and persuasion. See Grosholz 2007.

[17] Part of the prescriptive role of axiomatization is strictly speaking constitutive. Structures and techniques, concepts and performances, act as instruments and expressions of intended, and endorsed, standards of construction, ex., the determination of a subject matter through the implicit definition of terms, or concepts and the constitution of models and domains. This stipulative function may be considered an instance of the broader pragmatic dimension of formalism.

For example, the pragmatic value of axiomatics in mathematical practice (and as mathematical practice) concern several valuable functions: explicit formulation of theoretical statements and explicit, differentiated, representation of their proof; theory demarcation; theory evaluation; theory communication. Schlimm 2013.

partial truth or fuzzy predication at work in the treatment of linguistic cases. The duality of content is captured by two distinctions: a main distinction is between linguistic and pictorial contents, and a more specific one, between intrinsic and extrinsic pictorial contents.

(B3)-Moreover, the distinctions between both kinds of contents –that is, sets of categorizations or features–, linguistic and pictorial, intrinsic and extrinsic, raise the question of understanding accuracy of representation as a form of pictorial approximation without approximate truth. Is vagueness a form of approximation? Unlike accuracy of measurement or depiction, vagueness is only a form of approximation in fuzzy set models.

(B4)-Finally, pictorial representation and reasoning are inseparable; in ordinary and scientific cognition, representation –that is, representations and representing– often requires some modality of reasoning or computation and different modalities of visual thinking are common and valuable and require some form of pictorial representation too; then, representation without truth suggest a different kind of inference-making rules and procedures. If fuzzy representation involves some kind of fuzzy reasoning, it can also support its extended application beyond reasoning for the sake of representation to reasoning from representation, for instance, by means of a fuzzy calculus of pictorial representations. Rules must cover the related cognitive tasks that include inference, computation and problem-solving. But this is not without limitations for the role of truth and reasoning. Scientific processes of cognition may be formalized and controlled through rules and procedures connecting what I call intrinsic and extrinsic content. I argue that their formalizability, use and validity are restricted and the assumption of their general use is misguided.

In the rest of Part 1 of the paper, I can focus only on claims B1 and part of B2. In Part 2 I discuss further B2 in relation to B1, and extend the discussion to B3 and B4.

(1) Application of mathematics in the representation of images: from geometry to set theory.

It is worth noting that mathematics has long been applied to images, especially in a number of ways that rely on the quantitative categorization of visible properties. In addition to topological properties of curves and surfaces, metric standards contribute quantitative measures of size and distance. For instance, analytic geometry associates with the sets of points that form geometric curves algebraic equations and functions. The equivalence between both types of representations is established by the fact that they provide modes of coordination, a mapping, between two sets of real

numbers. To each value of a variable ranging over one set of real values represented by points on one line, or axis, corresponds the value of another on another axis. The curve is a set of points represented by the set of such pairs of numbers, their coordinates. Cartographic representation extends the coordination techniques to surfaces as a set of infinite diagram curves; although typically they only depict the ones corresponding to area boundaries such as coast lines and administrative borders. Coordinate systems in the form of two-dimensional grids are superimposed upon maps and provide a representation, an analysis, of each location (the pair of coordinates are a numerical mapping).

What about images? In previous work I have discussed how fuzzy set theory has been applied to the empirical (ex., Zadeh) and conceptual (ex., Smith) understanding of linguistic vagueness as quantitative, conceptually precise model of categorization practices.[18] The same formal approach has been applied to images, and this in two different ways I call synthetic and analytic.

The synthetic approach extends to images the same cognitive focus and the formal resources used to model vague predicates, with the categorization of a particular image by means of its inclusion in a set; inclusion in the set is represented by a degree of membership, a mapping to a value within the range [0,1]. Visual description and experience take meaningful place only within a perceptual system. From a categorization standpoint, it makes sense to claim that linguistic and visual indeterminacy and vagueness are inseparable.[19]

The analytic approach considers images as collections of individual points. These perceptual objects or information points are units of visual data to be categorized separately, in terms of a fixed set of perceptual properties.[20] What counts as a unit, e.g., a digital pixel, is typically established by material and design properties of the medium. On that basis, collective properties of the pixels can be then associated with additional, higher-level categorizations, intrinsic or extrinsic, constrained by the aims and categorization schema relevant to a particular context of pattern recognition –examples of problems involving the determination of extrinsic contents include facial recognition and medical diagnosis. Post-analysis synthetic approaches may deploy a number of composition strategies, beginning with the simplest computational assumptions such as set-membership, additivity

[18] Cat 2015b and forthcoming.

[19] Peters 2009 and Peters and Pal 2010.

[20] This line of conceptual and technological application to the case of images originates in works such as Rosenfeld 1979 and Pal 1982 and 1992.

and linearity.

(2) Cognitive framework of set-theoretic methodology of analysis and synthesis: categorization, classification and many faces of digital geometry.

Both approaches to so-called visual analysis have set-theoretic expression, based on the assumption that visual cognition takes place at the level of classes, not individuals.[21] The analytic approach has combined classical fuzziness with other generalizations of set theory such as rough and near sets.[22] A cursory examination of the terms of application of the set-theoretic formalism brings out problems and features that set the pictorial case apart from the linguistic alongside similarities.

Even the perceived relative homogeneity in individual images, or parts thereof, may be the outcome of a process of post-analysis synthesis, for instance, either by applying clustering algorithms to establish classes or statistical techniques for the calculation of collective features out of the units of analysis. Now, the methodological challenge consists in linking the application of the two approaches, especially with critical and informed attitudes in order to serve best specific aims of representation and reasoning at hand by the techniques and standards at hand.

Indeed, while set-theoretic approaches may be embedded within other formal frameworks, the latter have become the basis for classical algebraic alternatives.

Digital geometry is one such basic framework.[23] It is based on the manipulation of numerical algorithms for the sake of representing and reducing blur. The discrete structure of a pixelated image receives arithmetic representation in the form of matrices of values of binary variables. The central formal assumption is the model of an image by a multi-dimensional system of binary digits, typically two-dimensional arrays (mxn) for grayscale, and three-dimensional ones for color scale. The algorithms assume a central epistemic distinction, between the perceived or recorded blurred image, B, and the true or desired sharp image, X.[24] The form of the problem and the master reconstruction task of solving it are then to find the digital expression of the true or exact image from the blurred one. X= F[B], where F represents a variety of mappings and techniques. If F represents the so-called deblurring, its inverse represents blurring.

This task is constrained by another epistemic assumption, the

[21] Ibid., and Orlowska 1985.

[22] See below and Peters and Pal 2010.

[23] See Hansen, Nagy and O'Leary 2006 and Klette and Rosenfeld 2004.

[24] Hansen et al. 2006, ch. 1.

conceptual distinction between the exact blurred image, B_{ex}, and the error factor, E, random noise associated with the process of image/data production; one represents the source of imprecision, the other the source of inexactness. How is the distinction drawn? It requires both empirical knowledge of the instrument and statistical analysis. The recorded or perceived blurred image may then understood in terms of relations such as the simplest one based on additive linearity, $B= B_{ex} + E$. The fuzzy-set development of this kind of digital analysis and treatment must distinguish between error and uncertainty.

Within the analytic approach and with images modeled as sets of points of visual data, the formal modeling of digital technology no longer draws a distinction between image and picture, perception and picturing, anatomy and technology. In fact, the approach provides a more abstract and general framework; within it these are all different instances of the same kind of cognitive system.

A perceptual system is defined as a pair formed by a finite non-empty set of sample perceptual objects (standards or exemplars), O, and a non-empty countable set of probe functions, F, representing possible attributes or features of the objects.[25] Perceptual systems may be considered particular kinds of information systems, where the universe of objects is a set of data.

The relevant objects are sources of reflected light in the visual field. The may be the image pixels on a digital screen. The mathematical labeling of sensations provided by a so-called visual probe is a requirement on the operative notion of observability.[26] The role of visual probe functions is to map visual field of points of reflected light sources onto a set of real numbers, assigning feature values; this is the task of feature extraction. In this mapping, signal values are sensations connecting sense stimuli (e.g., in retinal cells) and brain activity (e.g., in visual cortex cells); then probe functions model seeing as transmission processes in perception. They are said to extract perceptual information. Probe functions can map a physical continuum in the environment onto a value distribution over a phenomenological continuum of sensations.[27]

A few details will show the formal framework of practices and

[25] Peters 2009.

[26] Peters and Pal 2010.

[27] A precedent in the appeal to similarity between sensations in a manifold can be found in Poincaré's empirical account of visual space and geometrical knowledge in his *La Science et l'Hypotèse* (1902). For Poincaré this has consequences for the foundation of mathematics; the phenomenological facts about similarity cannot underpin arithmetical notions of ordering and continuity. For example, the empirical basis fail to satisfy transitivity relations: A appears indistinguishable from B, B appears indistinguishable from C, yet A appears distinguishable from C.

concepts of visual categorization and representation that I am interpreting in *contrast* to the case of linguistic representation and reasoning. While in the linguistic case, some empirical, methodological and subjective dimensions of the categorization process are black-boxed, in the pictorial case, they find formal representation.

From a set-theoretic standpoint, the formal model of cognition at work here extends the application of the linguistic model: a concept or category is introduced extensionally, as a subset of the universe of objects; and to categorize is to assign set membership. The application reaches a higher level of abstraction in extending to the outcomes of cognition beyond perception. A family of concepts (subsets) constitutes abstract knowledge about the universe and the corresponding partitioning enables the classification of objects.[28] A knowledge base is a relational system K=(U, R), where U is a non-empty set of objects and R a family of equivalence relations on U partitioning the universe into concepts or subsets. A subset X is R-definable or R-exact if it is the union of basic categories, which are defined by indiscernibility relations on U. The classification is exact or precise if based on R-exact concepts. If they cannot be defined in the R knowledge base, they are R-rough or vague sets.[29] The exact definition of X in U signifies also the completeness of knowledge about U; then notions of entropy, for instance, provide a general measure of roughness.[30]

It is worth noting that the set-theoretic representation provides the operative conception of vagueness with several dimensions of objectivity: *formal*, *methodological* and *empirical*. First, the mathematical character of fuzziness –the symbolic relations that help form the relevant concepts– provides a dimension of *formal objectivity*. Second, the application of the formalism is constrained by a more or less broad context of standards, goals, rules, choices, etc. that provide a structure of *methodological objectivity*.[31] Third, as a semantic matter of truth, authors such as Smith locate the objectivity of the linguistic vagueness of a predicate in the properties the predicate represents in contrast with a property of the epistemic state of the language user (that is, in relation to the determinate property conceptualized according to the use of the predicate). Now, when the property of the individual in question, a pixel, is itself a perceptual property, with a physical basis at its source and its detection, the distinction between objective and epistemic conditions collapses. The set structure, through a membership

[28] Mushrif and Ray 2010, 10.2.
[29] Ibid., 10.3.
[30] Sen and Pal 2010, 3.6.
[31] For a discussion of the limits of this methodological objectivity see Cat 2015a.

function, models two *empirically objective* states: an act of categorization and the objective cognitive relation it categorizes, namely, a perceptual property between the medium that displays the image and the viewer such as brightness and the tonal contrast that define an edge.

(3) Conceptual resources in set-theoretic models of vagueness: fuzzy, rough and near sets.

The conclusions of my discussion of pictorial precision and certainty in informal philosophical analysis extend to the application of fuzzy formalism. In fact, the do more broadly; in image analysis fuzzy formalism has been applied as part of different generalizations of set theory: near sets, rough sets and fuzzy sets and their possible combinations.[32] These are all kinds of Cantor sets in which membership is established by a law that enables the grouping into a whole. This account is not a reductive, purely extensionalist view of sets and the categories associated with them.

So-called probe functions of different degrees of generality assign feature values (measures of strength) to members in a set associated with the feature in question. Operating as membership functions, they identify the set or feature through the description they provide (vector of feature values for each object).

Also near and rough sets provide general formal concepts for modeling vagueness.[33] In the technological context that motivated their introduction, vagueness is understood in the more cognitive of the objective senses, in terms of indefiniteness in the practice of classification decisions. Some authors distinguish between two kinds of indefiniteness or ambiguity: feature-ambiguity based on vague definition of region or set boundaries and feature-ambiguity based on indiscernibility.[34]

Fuzzy sets provide measures of closeness in membership and truth relative to a feature; rough and near sets provide formal measures of closeness in indiscernibility. Either way, the boundary is established as a matter of decidability of categorization. Both sets of notions may thus be linked to measures of similarity relative to a feature or category. But fuzziness is assessed in terms of degrees of membership based on comparative considerations of similarity.[35]

Nearness and roughness are, however, somewhat more general. Nearness is applied to subsets belonging in different sets or clusters;

[32] Sen and Pal 2010.
[33] Ibid.
[34] Ibid.
[35] Cat 2015a.

roughness is involves exact approximations and it extends to objects declared outside the set. Together these notions model complex forms of indefiniteness, indeterminacy or uncertainty, features that are often considered, mistakenly, equivalent to vagueness or ambiguity. From the formal standpoint, they may be all considered different forms of approximations (see part 2 of this paper).

Fuzzy sets are near each other if the fuzzy probe function assigning membership grades assign values to at least one object in each within a margin set by the tolerance range or distance e. This constant sets the margin of difference in degree of similarity or closeness of description, as in Smith's notion of vagueness, except this is a weaker shared version in respect of categorization, not truth. Nearness is one way to formalize the criterion $V_4(p)$.

In near sets, disjoint sets resemble each other: description matches correspond to observable similarities. Descriptions of members of near sets are partially matching.[36] Nearness measures the degree of perceived similarity, or, equivalently, of perceived difference between description values, between objects within each single set is higher than across (near) sets.

The synthetic dimension of image processing rests on the use of these general notions of sets: to the units of analysis within the image and to the classification of image as part of a larger class. For instance, a method of pattern recognition through perceptual tolerance identifies a perceptual system and introduces a tolerance relation –the relation defines a cover on the set of sample space of objects: X and Y disjoint subsets of O are tolerance near sets if and only if there exist respective subsets of X and Y that resemble each other relative to a subset of F and a tolerance class e, least angular distance so that all points indistinguishable from any x lie with distance e from it. The distance e may represent, for instance, pixel size as a measure of degree of resolution; each uniform pixel is a tolerance class.

Rough sets are characterized in topological terms by the size of the non-empty boundary region of their so-called approximation, or the difference between their upper and lower approximations.[37] Central to the approximations is the notion of partition of the universe into a quotient set of elementary equivalence classes of indiscernible objects with identical feature function values. Rough sets are roughly approximated, by two exact sets corresponding to two exact concepts: these equivalence classes among their subsets and intersecting sets, with a lower approximation in the union

[36] Peters 2007.
[37] Pawlak 1981.

of such subsets within and an upper approximation in the union of intersecting sets with outside. If they are not identical to their lower approximations and the upper boundary provides a non-empty precisely determined boundary region; these enable segmentations into non-overlapping homogeneous regions. The measure of roughness at the boundary r=1-(cardinality of lower approx./cardinality of upper approx.) a measure of inexactness or incompleteness of knowledge about X

The notion of rough boundary plays a formal role similar to that of the more general notion of tolerance relation in near sets. In image analysis the objects are the points or pixels into which digital images are analyzed. Each approach provides the basis for establishing clustering algorithms that enable the recognition or classification of images, for instance, the rough-fuzzy c-means algorithm.[38]

Fuzzy sets enter the synthetic process of image treatment through algorithms such as the fuzzy c-means algorithm. An overlapping partition is typically fuzzy, handled by fuzzy membership functions. In a trivial and broadest sense, a fuzzy boundary is also a fuzzy overlapping partition with its complementary, or union of complementaries.

Another philosophical lesson follows at the formal level: As in the linguistic case, image classification is a holistic, relational affair. Fuzzy membership functions take the form of a matrix, m_{ij}, for n objects and l centroids or potential clusters. Relational, comparative talk of classification replaces monadic talk of categorization. Classification and class definition and portioning are based, in fuzzy sets, on the contextual and constrained selection of prototypes (centroids); the latter form the subset of the classified objects that help assign weights or feature values, both in the range set by maxima and minima and the centering averages/means.

Successful clustering procedures yield different measures of feature value distances (or degree of inhomogeneity or variation) between clusters (centroids) larger than distances within clusters. Each centroid may help establish a cluster of objects in terms of the minimization of a so-called objective overall multi-cluster function: in terms of a normed distance or difference in feature value weighed by membership values for different centroids. Each object gets membership values relative to each centroid or cluster to degrees constrained by the minimization of the objective function.[39] Classification or categorization involves the application of a

[38] Maji and Pal 2010.

[39] The algorithm consists in iterations of updating recalculation of the initial random selection of the means associated with each centroid feature value in terms of the given membership values until a iteration leads to stability of values, the c-means, with recalculated membership

unitary fuzzy membership function over the range of centered distances. The feature value of the centroids is then the means of the feature value weighed by membership values.

Rough clusters may be formed out of fuzzy ones. One driving motivation is the intuition that the definition of the class is vague or uncertain to a degree allegedly best modeled by the upper approximation of a cluster as a rough set.[40] Indistinguishability models approximation. Accuracy, a, is modeled by the ratio of the size of the lower to the upper approximation extensions, leaving fuzzy membership to model overlapping partitions. Roughness is then modeled in terms of inaccuracy (1-a). This receives the additional epistemic interpretation of knowledge completeness about the universe. Notice that this is also a model of indeterminacy in terms of equivalence and redundancy or overrepresentation, even possibly underdetermination, but not necessarily vagueness. Vagueness would be better modeled still by fuzzy membership and degrees of truth or at least some weaker semantic notion suitable for pictorial success.

In rough sets, fuzzy membership becomes relevant in the upper approximation of a cluster: only boundary objects are fuzzified; there fuzziness models approximation.[41] Rough sets assign the objects in the lower approximation a membership value of 1; when the difference with the next lower membership reaches above a threshold value, their cluster membership is declared exclusive (a second contextual value, data-dependent, empirically determined by the average highest membership value difference over all objects).[42]

The cluster, then, has a crisp core and a fuzzy boundary around a weighed average. The partition yields a classificatory segmentation, a method of categorizing analysis of image pixels. Distributions of values of some feature must be correlated with their perceived spatial distributions (static or dynamic).

(4) Analytic and synthetic forms of vague categorization.

Most quantities or measures of interest modeling some form of uncertainty correspond to the intrinsic kind. Fuzzy image analysis is based on the premise that the properties of edge, boundary region or relation in images are not generally represented in sharp terms.[43] Some such properties are

values showing an increase above a given threshold (contextual choice).

[40] Maji and Pal 2010, 2.2.

[41] Ibid., 2.5. More on approximation in the follow-up Part 2 of this paper.

[42] Ibid., 2.7.

[43] Pal and King 1980 and 1981.

determined more readily than others, but their choice is constrained by the aims and standards of the analysis or processing –ex. as formulated by performance indices (see below). They constitute the feature vector that represents the unit of information or perception such as a pixel. One basic feature is tone scale value –as simple scalar or as vector in basic color space of 2, 3 or 4 dimensions.

The collective form of synthetic representation consists in derived measures of data populations are first and second-order statistics determining the resolution and precision of images. Examples of first-order statistics features are comparative neighborhood relations of homogeneity and edge value; examples of higher-order ones are Haralick's texture featured based on the spatial distributions of perceptual units measured by joint probability distributions, from angular moments to sum and difference entropies.[44]

The basic, simplest model image for developing the generalized set-theoretic approach is the greyscale digital image. The image is identified with the intrinsic content of a region in the digital medium, the picture, and consists in an arrange of grey values presented by the different individual pixels associated with varying degrees of light intensity on a continuous scale ranging from white, the brightest, to black, the darkest. The different levels form the universe that is represented in a histogram. Ambiguity or vagueness is introduced as the uncertainty or incompleteness of knowledge about this universe. The goal is to provide an objective formal measure of grey ambiguity in the arrays and a technique for reducing it replacing granulation with segmentation.

The values on the in-principle continuous scale enable a gray value gradation over an array of contiguous pixels. This is the property within or between dark regions that mid-19th-century photographers valued as the half-tones on which much richness of detail and realism depended. But vagueness should not be confused with the gradation itself, which mathematically can be modeled otherwise, for instance, by gradients. The phenomenology of distributed values is equally what I explained above cannot identify the representational property of blur.

Instead, the tonal vagueness known as greyness ambiguity is a feature (category) of a boundary. That is, it's a feature of yet another, topological, feature that is in turn identified alongside a topological region, with a fixed finite area and geometric shape. However, the concepts of pure geometry only help categorize our experiences. In the phenomenological

[44] Maji and Pal 2010.

context of images, we deal with empirical geometry, the sensible categorization of experience. The boundary line that distinguishes two visible regions has an equally visible empirical identity in the form of a sharp contrast between homogeneous tone intensities, such as black and white, dark and bright. But the grey value gradation prevents the appearance of a sharp enough contrast, a boundary location. This is the so-called greyness ambiguity of the boundary that other visual cues lead the viewer to expect but cannot locate.

As a matter of objective possibility, not just incompleteness or uncertainty, the indefinite boundary can lie anywhere in the gradation.[45] To measure of the ambiguity, set theory offers two formal criteria: (1) vagueness due to fuzziness of the boundary (the feature associated with the application of a category for which an instance is available) and (2) rough resemblance due to indiscernibility between gray levels responsible for granulation. The construction of specific rough-set criteria such as information entropy measures is constrained by desiderata on formal representation such as non-negativity, continuity, sharpness, maximality, resolution, symmetry, monotonicity and concavity.[46] Some are formal constraints, others are formal but also motivated empirically, for instance, in terms of feature structure and maximum and minimum conditions.

As a synthetic cognitive phenomenon, the indefiniteness, vagueness or undecidability, to use terms authors take to be equivalent, can be traced to the vagueness of greyness level in individual pixels, even when the ambiguity is due to the indiscernibility between contiguous grey levels. The analysis applies also to the aggregate collective account of greyness levels represented in a histogram, where the regions are identified and aimed to be distinguished further.

Rough sets, in addition to fuzzy ones, help represent the homogeneous element identified in the phenomenology of blur in unaided perception and its technological extension. The interpretation of blur in terms of overrepresentation can be then understood in terms of either kind of overcategorization (relative to empirical or theoretical assumptions or background knowledge).

A number of related but different segmentation methods such as clustering and multilevel thresholding have been introduced in order to control grey ambiguity; they are in effect techniques for minimizing the adopted measure and differentiating grey levels into separate regions with more homogeneous contrasting levels, based on pre-set target threshold

[45] Sen and Pal 2010, 3.1.
[46] Ibid., 3.9-10.

values. The simplifying aim is to increase precision of features such as boundaries by increasing simultaneously homogeneity and disjointedness in sets of values. This kind of operation is known as feature-extraction. On the greylevel histogram, segmented regions are expected to correspond to regions in the vague image. Pictorial desiderata in intrinsic content of the image are often based on background knowledge that helps assign the features of extrinsic content. It relies on considerations of resemblance. For instance, as in the case of blur, theoretical or perceptual habits lead to an expectation of a recognizably sharp visual boundary.

(5) From intrinsic to extrinsic vague categorization and content.

Above I have focused on the treatment of intrinsic image features. In the process, I have also noted that aspects of their categorization such as ambiguity and of their control such as threshold values are associated with extrinsic features. In fact, the representational function of pictures is determined by the accuracy of their extrinsic content. This is the dual problem of *content development* and *content coordination*.

Content development builds originally on intrinsic content, the recognized features in intrinsic categorization of the picture; but to do so requires a coordination rule that links extrinsic to intrinsic content, IC-EC rule. To the extent that the IC-EC rule provides a mode of inference, we can claim that it enhances (reasoning-based) representation through (representation-based) reasoning. Here I don't distinguish between reasoning, inference and computation. The relevance of specific theoretical, empirical or conventional interpretations of the IC-EC rule depends on the context of production of the image. Out of IC-EC links one can apply further links of the form EC-EC', and so on.

For medical diagnostic purposes, one such rule is an explanatory hypothesis with empirical support positing a causal link between certain types of breast cancer and sudden increases and asymmetries of local temperature distributions in thermograms. We see tonal distributions, but with the association encoded in the rule the intrinsic information is categorized further in terms of the recognition of cancer.

Another type of IC-EC rule for addressing content development takes the form of a compositional strategy. When only external content of the parts is available, the constructive strategy of the extrinsic content of the whole involves an IC-EC rule –or an EC-EC' one– that depends on the coordination of subunits of visual information through their individual partial IC-EC rules. These rules satisfy the following compositionality condition: for each unit i, $(IC\text{-}EC)_i=(IC_i\text{-}EC_i)$. They are applied along with auxiliary coordinating

techniques that involve stitching rules and constraints based, for instance, on features of the medium of representation and a code of conventions.

Galileo's drawings of the moon already follow this strategy, since the field of view of his telescope couldn't accommodate the entire image of the moon's bright side; a more modern example is the recent construction of a map the dark side of the moon. In the medical sciences, the computational source of brain imaging, especially functional-MRI imaging, is currently a popular instance of this synthetic process. At the very basic level of MRI imaging, the reliance on statistical calculations incorporating information from populations of similar individuals and statistical criteria leads to a controversial relation between image production and image interpretation and trust.[47]

Conversely, solving a problem of content coordination may contribute to solving the problem of establishing a picture's intrinsic content. In a given context of picture-making or interpreting, the identification of the extrinsic content and an IC-EC link help determine by selection the recognized and processing of properties that make up the relevant intrinsic content. A synthetic strategy, then, may precede and inform the task of analysis. Compositional strategies may involve the role of such top-down constraints.

Instances go back to the development of 15th-century mechanical, step-by-step, rule-based drawing techniques: the orthogonal grid structure in the arrangement of glass panes in windows became adopted as a tool for picture-making by means of mapping by geometrical analysis into smaller areas. The perception of the whole determines the representational content of the parts (top-down analysis), while the ensuing representation of the parts will, in turn, yield an adequate representation of the whole (bottom-up synthesis). The standard of adequacy of representation there is the kind of perceptual realism set by the experience of looking out a window, which the drawing aims to reproduce through the geometry of grid (in lieu of window) and the rules of classical perspective. This procedure is an instrumental, constructive reversal of the Sober-Fodor image criterion. Modern pixelation is the digital limiting version of the classical grid-based representational technique.

From a methodological standpoint, content coordination serves the epistemic function of evidence. This is analogous to the case of hypotheses supported by empirical data provided one can assume an auxiliary hypothesis linking the relevant information or concepts in the data, e.g., running a fever

[47] Bogen 2002 and Machéry 2014.

when the thermometer detects a temperature of at least 100.4F.[48] As my note on data, above indicates, however, the evidentiary value of data is a complex and contextual matter that can hardly be reduced to a general formal rule.[49]

The relevant result is that content development is a form of image interpretation by categorization; and thus it may present some form of vagueness. Image processing, however, is typically based on the auxiliary assumption that content development is based on content control through the minimization of vagueness. The alternative I have mentioned is the increase in blurring for practical purposes (for instance, legal protection of information). Behind the reduction lies a specific form of IC-EC rule linking the precision status, P or I, of intrinsic and extrinsic contents, namely, sub-rules PIC-PEC and IIC-IEC: they assume that the precision status of each type of content is linked to that of the other in a relation of identity. Only definite or precise intrinsic content guarantees precise extrinsic content; at the very least, it's assumed, intrinsic precision sets an upper limit to the possible vagueness in extrinsic content; since intrinsic vagueness only yields a measure of additional extrinsic vagueness.

Criticism from early photographic realists and the methodology for image analysis suggests that the formulation of the sub-rules is general or that their application is systematic.

But the PIC-PEC link, when in fact eliminating uncertainty in the treatment of the image, may be wrongheaded as a general strategy; it may be cognitively counterproductive. But the degree of precision of one level of content does not necessarily or sufficiently determine that of another's. Their dependence may go either way and is bound by context. Here are some scenarios:

(a) The most common case is the basis for the need of specific, but not universally applicable, IC-EC links; in the absence of additional constraints, precise intrinsic content –what the viewer sees– cannot fix the categorizations in the extrinsic content –what he sees in it. It cannot fix ambiguity, before the additional challenge arises from underdetermination of a set of possible precise extrinsic categorizations –whether the dot on the screen "is" a star or a tumor, a person, etc. Even if we leave aside indexicality challenges concerning the reliable determination of reference, the general form of the schema for IC-EC rules that enable pattern recognition must accommodate the fuzziness in extrinsic categorization. Each context of image production and interpretation might suggest different specific

[48] The role of this kind of auxiliary hypotheses and their status as a vehicle for values and biases is discussed in Longino 1998.

[49] Cat forthcoming, Part 1; Leonelli 2013.

instantiations with empirical or heuristic value.

(b) In a common type of case, blur of image either in perception or depiction is identified in relation to the standard or expectation of a precise property, such as a tonal or geometrical arrangement. That expectation may be suggested by habit, also by a contextual interpretive assumption about a corresponding sharp property in an object of depiction (extrinsic content). Precise extrinsic content can be then a condition for the ambiguity in the categorization that establishes intrinsic content.

(c) Another type of case is the one the Victorian perceptual realists and subsequent cognitive psychologists have established, that a variety of extrinsic categorizations rely on an intrinsic perceptual cue involving fuzziness. Determinations of distance, focus of attention, texture, volume, even the recognition significant structural detail depend on the absence of sharp contrast and the homogeneous segmentation supporting it. It might not, of course, be sufficient, and, as cases of type (a) show, a context of constraints is required. Nevertheless, it suggests a cautionary note against identifying the sharp/fuzzy dichotomy with the signal/noise one. From a cognitive standpoint, information-based entropy measures are no reliable guide to salient content.

The role of context suggests the incompleteness of any automated procedure. The three scenarios just outlined involve a motivated choice of what counts as the effective subset of the knowledge base, the range of effective conceived possibilities and connections. Constraints that make possible different degrees of indeterminate representation, additional uses of reasoning and the evaluation of outcomes depend on additional inputs from local constraints and external operation. Self-setting automated system cannot anticipate all the relevant constraints.

2 Conclusions

In this two-part paper I take up a problem in philosophy, vagueness, and tracked one particular solution into the world of mathematical science, set-theoretic fuzziness. My interest in this problem is twofold.[50] There is the philosophical problem itself, and then there is the problem's role of laying down lines of communication across divides between philosophical and scientific practices. From a disciplinary distance, these are practices that can share similarly formal and conceptual preoccupations, also an empirical interest in actual human practices such as representation and cognition. The philosophical problem emerges out of an interest in semantic practices in the

[50] I have expressed similar views in relation to causation in Cat 2006.

use of symbolic language, especially in representation and reasoning. Science offers its own symbolic, mathematical, practices to represent vagueness and to apply it in empirical representation of empirical phenomena. Here is a case in which science, mathematics, solves a problem for philosophy and philosophy retrieves it back for further analysis and use. Science doesn't conflict with philosophy, nor does it replace it.

What is this further use? I have argued that the further philosophical exploration of the scientific account yields a triple result in the form of inseparable solutions to three general problems: vagueness, pictorial representation and the application of mathematical in scientific practice.

Specifically, understanding the nature of the solution in terms of categorization provides the opportunity to extend the problem to the domain of pictorial practices and, again, to consider a similar solution from science. I have argued also that to understand the scope of that solution involves understanding scientific as well as artistic practices and cognitive activities. Vagueness in practice finds its rich contextual significance attached to diverse sets of activities, values and purposes. Scientific practices of empirical representation and technological control appear inseparable. In addition, the application of fuzziness to understanding and controlling categorization and reasoning is itself an instance of a practice of categorization and reasoning in the application of mathematics. To accommodate the full extent of the scientific solution to the problems of linguistic and pictorial in terms of categorization, I argued that it helps adopting more encompassing notions of categorization and empirical objectivity.

Within this extended framework, the case of pictures shows distinctive complications and challenges, also opportunities; for instance, in the philosophical and scientific understanding and using vagueness and pictorial representation. The practices of pictorial representation and reasoning present themselves as growth opportunities for scientific representation and control, especially if they are modeled and developed outside the boundaries set by the semantic standard of symbolic truth.

In relation to philosophical opportunities: I want to point out, again, that in my discussion I have assumed that representation is not to be understood or intended exclusively in terms of a single criterion such as categorization (and the role of similarity judgments in the process of its production and application in representation and thinking); although the discussion focuses on the relation between vagueness and representation according to the role of visual categorization. I have assumed too that the value and use of pictures is not reduced to representation or depending on a

role for representation. Exploring what vagueness can be and how outside the boundaries set by the following fours standards: (1) linguistic symbols, (2) the representational role of pictures, (3) the categorization and resemblance approach to representation and (4) the intrinsic/extrinsic content dichotomy.

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