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Some uncertain reflections on uncertainty

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Abstract This paper's goal is very simple, that of trying to look at the broad and multifaceted concept of uncertainty from a typically scientific point of view, namely, that consisting in representing concepts through numerical quantities. For the goal, it is supposed that the universe of discourse is amorphe (not previously structured), and that the basic treats of the meaning shown by the mother-predicate 'uncertain', from which it comes the concept of uncertainty, can be empirically captured in, at least, for what concerns its variability between the elements of the universe of discourse. Once 'measures of uncertainty' are defined in the same way as measures of length, area, volume, randomness, electrical intensity, etc., were introduced, there are analyzed the two cases in which 'uncertain' is confused with 'probable' and 'possible', to see which restrictions they mean in the use of 'uncertain'. What it is just attempted is to open a window towards a new, and more general way, of considering the term 'uncertainty'.

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1 Introduction

It is only recently that the old concept of uncertainty began to be reconsidered in itself and newly [1], but without being necessarily linked with probability either objective or subjective, or with statistics. Perhaps, one of the most salient and relatively new complements to probability in the treatment of uncertainty is Possibility Theory [2], and it seems today that both probability and possibility measures, either of crisp or fuzzy events [3], are to be taken into account for such treatment. In particular, possibility measures seem especially adequate when dealing with information that is imprecise and not susceptible to repetition, and except in some degenerate cases [4] they are not reducible to probabilities.

A look at the internet shows that the available information on the word 'uncertainty' appears in close relationship with very different and broad subjects like, for instance, Quantum Physics, Business, Actuarial Science, Weather Forecasting, Economics, Margin of Error in the Measurements, Politics, etc., and that its current study is mainly conducted through probability models fed by statistical data, even if not always this kind of data is fully available.

Uncertainty is pervasive in most of the fields of science and technology (as it is also in real life and ordinary thinking), and in all cases it seems that what really matters and worries is the quantification of the 'amount' of uncertainty in some sense attributable to the considered statements, or events, be them possibly submitted to repetition (random events) or not (singular events). All that is done, in general, under a very unclear concept of what the word uncertainty could actually mean, but it is almost always done by aiming at the measurement of the variables affected, in very different contexts, by some kind of uncertainty.

Nevertheless, up to now no attention seems to be placed on the concrete uses of the mother-predicate *U* = *uncertain* on general enough, and not previously 'structured', contexts. It is something like to have been considering the concept of a probability without some previous attention and some mathematical study of what the term 'probable' can mean as, for instance, George Boole and many others tried to do before the Kolmogorov's mathematical formalization of the probability concept; or to consider today temperature's numerical concept without previously take into account what historically conducted to both the study of the transmission of heat, and the building of thermometers. To capture the meaning of a statement concerning a predicate, like it is '*x* is *U*', some previous knowledge on its use in different contexts is necessary; without this kind of information, and even

if it is of a poor quality, no step towards a 'formalization', can be actually done. This is a basic principle for any trial of scientific thinking.
What this paper tries to do is just a reflection towards a way for posing how to use the predicate *U* = *uncertain* in a very general and perhaps clarifying form that, notwithstanding, could allow looking at *U* as a numerically measurable quantity. This reflection is based on comparing which differences both hyponyms but not synonyms predicates of *U, P* = *probable*, and ***P*** = *possible* , can show with the predicate *U*; something that seems to be necessary and previous for a more advanced or complete study on the concept of uncertainty in itself, and its measuring in particular.

2 On the concept of uncertainty and the predicate uncertain

It seems difficult, if not impossible, to actually say something that could be 'fertile' [5] on the abstract and debatable concept of uncertainty, without at least knowing how the (usually imprecise) predicate *U* is used in any not previously structured universe of discourse *X*, and saying it in a scientific-like form. That is, without trying to introduce some 'exact thinking' [5] on the subject, as it can come, for instance, by treating U as a quantity in such a way that the corresponding statements '*x is U*', for *x* in *X*, could be submitted to some numerical control through a measure of the 'extent' up to which each *x is U*, or *x* shows uncertainty, that is, shows the property named by *U*. Like it is done in the cases of length, surface, volume, random events, electrical intensity, etc.
Notice that without referring to some universe of discourse, *U* alone is for nothing; *U* is just a name for a property to be 'potentially' shown by some objects x on which the meaning, or use, of *U* depends through the elemental statements '*x is U*'. U only gets a meaning once applied to some universe of discourse. For instance, the idea behind the quantity 'area' only did appear after the measuring of some surfaces was necessary. Names only actually appear and are more or less fixed in language, once they are firstly applied to some objects for distinguishing some property attributable to them. Without previously capturing (even by simple perception) if given *x* and *y*, there is, or is not, one of them that exhibits less or more than the other the property named by *U*, no quantitative measure of the extent up to which the elements on *X* show *U*, is actually possible.

2.1

To begin with a reflection, let's try to follow it up by means of some of the ideas presented in [6] for interpreting the predicate *U* as a quantity in the set

X to which it is applied. To such an end, it will be supposed that two relations $\preccurlyeq_U \subseteq X \times X$, and $=_U \subseteq X \times X$, respectively translating the either thought or perceptively captured empirical relational statements that follow, are known,

- x is less U than $y \Leftrightarrow x \preccurlyeq_U y$,
- x is equally U than $y \Leftrightarrow x =_U y$,

for x, y in X, and without whose acquaintance with it cannot be clearly understood how the predicate U is used in X; and less again how it can be treated as a numerically measurable quantity. Let's work under the two hypotheses:

1. $\preccurlyeq_U$ is a preorder. That is, $\preccurlyeq_U$ is a reflexive and transitive binary relation in X.
2. $=_U = \preccurlyeq_U \cap \preccurlyeq_U^{-1}$, with $\preccurlyeq_U^{-1}$, the symmetric relation of $\preccurlyeq_U$, defined by '$x \preccurlyeq_U^{-1} y \Leftrightarrow y \preccurlyeq_U x$'. Hence, $=_U$ is the equivalence associated with the preorder $\preccurlyeq_U$.

It is neither sure that in all the cases the relation $\preccurlyeq_U$ is a preorder, nor that $=_U$ is always its associated equivalence classifying X in classes of equally uncertain elements. Nevertheless, and just for the sake of a first reflection like the one tried in this paper, it is not a bad at all option to work under some hypotheses allowing to pose in mathematical terms the problem of studying U. At the end, without accepting that U 'varies' in X and by knowing how it happens, it has no sense to attribute a numerical quantity to U.
Notice that it does not seem that a general enough description of the behavior of U, could be captured by an if-and-only-if definition, like it is, for instance, with the predicate 'prime' in the set of positive integers. In general, the behavior of U is only describable through some rules linking the elements x with the statements 'x is U', like it is done, for instance, with the predicate 'small' in the interval [0, 10] of the real line, whose meaning could be captured by the three basic rules '0 is small', '10 is not small', and 'If x is small and $y < x$, then y is small'. That is, by trying to capture, from the actual uses of the predicate, a kind of 'organization', or elemental structure, among the elements in the set X in which U is used.
Discourses on X are often done for trying to clarify what can be asserted on its elements x through some kind of 'ordering' among these elements; that is, by introducing in X some organization, hierarchy, or structure that, in the current case, should be related with how the predicate 'uncertain' is actually

used in X. Hence, the graph $(X, \leqslant_U)$ represents the organization the current use of U introduces in the universe of discourse X, and it can be called the *elemental meaning* of U in X.

2.2

Let's suppose again the two, and supplementary, prototype-hypotheses,

3. There is at least an element x_0 in X such that it is '$x_0 \leqslant_U x$',
4. There is at least an element x_1 in X such that it is '$x \leqslant_U x_1$',

for all x in X. If there exists a mapping m_U: $X \longrightarrow [0, 1]$, such that it will be said that m_U is a measure of U in X. It should be understood that $m_U(x)$ tries to reflect the 'extent' of the 'amount' up to which x is U, and that property (a) is crucial for this goal. If, for instance, and once it is $x \leqslant_U y$, it were not fixed which of the numbers $m_U(x)$ and $m_U(y)$ is the greatest of both -that is, 'how' m_U grows- m_U will be useless: For what could be useful a volume's measure under which to a cube greater than a given one, it were impossible to state which has more cubic meters? Under all that is how the triplet $(X, \leqslant_U, m_U)$ can be considered a quantity, and m_U the 'ideal' membership function of the fuzzy set in X with linguistic label U.

Remark Provided X is endowed with a binary operation 'v' allowing to 'add', or to 'compose' its elements, as well as a way to know if two elements in X are, or are not, 'incompatible', a function m: $X \longrightarrow R^+$, and provided x and y are incompatible, can be:

- sub-additive, if $m(x \vee y) \leqslant m(x) + m(y)$: Some 'extent' is lost when passing from incompatible x and y, to their composition $x \vee y$;
- super-additive, if $m(x \vee y) \geqslant m(x) + m(y)$: Some 'extent' is gained when passing from incompatible x and y to $x \vee y$;
- additive, if m is both sub and super-additive, or $m(x \vee y) = m(x) + m(y)$: No 'extent' is either lost, or gained, but preserved, when passing from incompatible x and y to $x \vee y$;
- neither sub, nor super-additive: It is not possible to foresee, for any incompatible pair x, y, if some 'extent' is lost, gained, or preserved, when passing from x and y to $x \vee y$.

In the case X is a lattice with minimum 0, conjunction ($\cdot$), and disjunction ($+$), with $+ = \vee$, and defining incompatibility by $x \cdot y = 0$, last definitions are

those that are typical in mathematics. A nice example of measures that can be always either sub-additive, or super-additive, or additive, is given by the so-called lambda-measures [8].

2.3

Since in most cases, and often due to the cardinality of X, it is very difficult to completely know the full graph $(X, \leqslant_U)$, the working scientist is so frequently limited to directly proceed through a function m approaching m_U (the membership function of the fuzzy set labeled U), defined on the basis of what is partially known on $\leqslant_U$, that is, on how to compare the statements 'x is U', a knowledge that could be easily non coincidental with that furnished by $\leqslant_U$, and that, at its turn, shows the relation $\leqslant m \subseteq X \times X$, defined by,

$$x \leqslant m \;\; y <=> m(x) \leqslant m(y).$$

This new relation $\leqslant_m$ is larger than the older $\leqslant_U$, since it can be supposed,

$$x \leqslant_U y \;\; => \;\; m(x) \leqslant m(y) \Leftrightarrow x \leqslant_m y,$$

that is, $\leqslant_U \subseteq \leqslant_m$; thus, m 'adds' to $\leqslant_U$ the difference-set $\leqslant_m - \leqslant_U$, and the new graph $(X, \leqslant_m)$ contains, in principle, more arcs than the older $(X, \leqslant_U)$. Both graphs only can be coincidental if the set $\leqslant_m - \leqslant_m$ is empty, in which case it is said that *m perfectly reflects* the use of U in X [8].
As the values $m(x)$ of m belong to the totally ordered unit interval $([0,1], , \leqslant)$, for all pairs x, y of elements in X it is always either $m(x) \leqslant m(y)$, or $m(x) > m(y)$, and the relation $\leqslant_m$ (obviously a preorder) verifies, for all x, y in X,

$$\text{either } x \leqslant_m y, \text{ or } y \leqslant_m x,$$

that is, $\leqslant_m$ is a *total* or *linear* relation, and, hence, the identity $\leqslant_m = \leqslant_U$ only happens in some of those particular and perhaps rare cases in which $\leqslant_U$ is, at its turn, a linear binary relation. Thus, the let's say, working-meaning $(X, \leqslant_m)$ is usually too large.
Consequently, anyone who only can work with a numerical function like m, should be conscious of adding more 'meaning' to U than that U itself actually has. Of course, the same happens if the work is done with the 'ideal' measure m_U, that can or cannot perfectly reflect the meaning of U in X. Of course, the relationship between m and m_U depends on the kind of (usually unknown) approximation m is of m_U.

Remark In the next two sections there will be considered, although in a superficial form, the 'restrictions' that, in both Probability and Possibility theories, are imposed to the predicate U once it is respectively identified with the predicates P = probable, and $\boldsymbol{P}$ = possible. These identifications require that the universe X is endowed, respectively, with the algebraic structures of Boolean algebra in the probabilistic classical case, Orthomodular Lattice in the Quantum one, and De Morgan algebra in the possibilistic case.
It should be noticed that neither numerical, nor fuzzy probabilities of fuzzy events [9] are taken into account in this paper.

3 On the probabilistic interpretation of 'uncertain'

In the field of classical probability theory there seems to exist the tendency to identify the predicate U with the predicate P = Probable, by additionally pre-supposing that the set of statements $X(P) = \{x \text{ is } P; x \text{ in } X\}$ is isomorphic to a Boolean algebra endowed with the three classical operations of conjunction (·), disjunction (+), and complement ('), respectively translating the linguistic conjunctives 'and', 'or', and 'not' [9]. The partial ordering of the Boolean algebra is the binary relation $\leqslant$ defined by $x \leqslant y \Leftrightarrow x \cdot y = x \Leftrightarrow x + y = y$, also translating the material implication $x \longrightarrow y = x' + y = 1$, representing the 'true' conditional statement 'If x, then y'. Of course, the algebra contains a minimum element 0, and a maximum element 1, with which it is $0 \leqslant x \leqslant 1$, for all x in X.
Probabilists also seem to identify the relation $\leqslant_P$, generated by the use of P in X, with the partial order $\leqslant$ of the Boolean algebra [10]; that is, they seem to suppose that,

$$'x \text{ is less } P \text{ than } y' \Leftrightarrow x \leqslant y \Leftrightarrow x \longrightarrow y = 1 :: \leqslant_P = \leqslant,$$

even if the only that, in general, seems to be for sure is the inclusion $\leqslant \subseteq \leqslant_P$. In this way, the elemental meaning of P is, in principle, enlarged with the difference-set $\leqslant_P - \leqslant$.
Concerning the relationship between U and P, it should be noticed that the only that can be generally stated is: 'If x is P, then x is U', but not always the reciprocal 'x is U' => 'x is P'; in general, both statements are not equivalent. Hence, it seems that it can be also supposed not only such conditional statement, but also that

x is *less P than* y => *x* is *less U than* y :: $\leqslant_P \subseteq \leqslant_U$.

In this way, probabilists 'reduce', in Boolean algebras, the 'meaning' of *U* to that of *P* and this relation to the partial Boolean order $\leqslant$, whose coincidence with $\leqslant_P$ is just supposed. Notice that the only that is for sure is $\leqslant \subseteq \leqslant_P \subseteq \leqslant_U$, and that although $\leqslant$ enjoys anti-symmetry ($x \leqslant y$ & $y \leqslant x$ => $x = y$), $\leqslant_P$ only verifies ($x \leqslant_P y$ & $y \leqslant_P x$ => $x =_P y$) that does not always mean $x = y$.

Additionally, probabilists measure the 'extent' up to which *x* is *P*, by means of probabilities, or mappings $p: X \longrightarrow [0, 1]$ verifying [9],

1) p (1) =1, and
2) If $x \cdot y = 0$, then $p(x + y) = p(x) + p(y)$,

from which it easily follows p (x') = 1-p (x), p (0) = 0, and '$x \leqslant y$ => p (x) $\leqslant p$ (y)'. That is, probabilities *p* are a particular kind of 'additive' measures for the predicate *P*, provided it is interpreted by the supposed 'meaning' $\leqslant_P = \leqslant$; their numerical values only grow by adding the measures of the disjoint component parts of the events. Notice that if $\leqslant_P - \leqslant$ is not empty, it cannot be always stated that '$x \leqslant_P y$ => $p(x) \leqslant p(y)$'; hence, it cannot be stated that *p* is a measure relative to $\leqslant_P$, but only to $\leqslant$.
In the Kolmogorov's formalization of probability, the algebra *X* is taken as a Boolean sub-algebra [9] of sets in a set *Y*, that is, *X* is a 'field' of events, a part of $P(Y)$ whose elements can be considered representing 'repetitive' or random events. Anyway, and thanks to the Stone's Representation Theorem, Boolean algebras can be always considered as isomorphic to a 'field'.
Of course, like with the predicate *U*, the working-meaning $(X, \leqslant_p)$ given by a probability, is either larger, or not comparable with, the (empirical) elemental meaning $(X, \leqslant_P)$, and what is now supposed is that $\leqslant_P = \leqslant \subseteq \leqslant_p$. Not only all this means a 'reduction' of *U* to *P*, and of the relation $\leqslant_U$ to the partial order $\leqslant$, but also *p* 'reduces' the measures for *U* to the sub-class of additive measures for *P*, once the hard supposition that the universe is a Boolean algebra is accepted. Actually, that probabilistic view on what 'uncertain' is, actually coarsens the predicate *U* introduced in section 1; in particular it is difficult to believe that m_U can be always additive, even in the hypothesis that an operation $\vee$ allowing to represent a suitable form of 'composing' elements could exist.

Remarks 1) In the case of Quantum Probabilities on Orthomodular lattices [11], the classic 'equivalence' existing between $x \cdot y = 0$ and $x \leqslant y'$ (that in Boolean Algebras holds thanks to the von Neumann Law of 'Perfect Repartition', $x = x \cdot y + x \cdot y'$, for all x, y), is reduced to its half part "$x \leqslant y$' => $x \cdot y = 0$". Anyway, in this case, the argument with respect to U is almost the same: $\leqslant_P$ is made coincidental with $\leqslant$, the partial order of the lattice.
The only difference with the classic case lies in that axiom 2 of probabilities is often changed to: If $x \leqslant y'$, then $p(x + y) = p(x) + p(y)$, by taking 'contradictory' instead of 'incompatible', and implying 2 in the case the Orthomodular lattice is just a Boolean algebra, in which case incompatibility ($x \cdot y$=0), and contradiction ($x \leqslant y$'), are formally indistinguishable.

2) The hypothesis that the universe X is endowed with a lattice structure (Boolean, Orthomodular, De Morgan, or whatsoever) is not a kind of 'natural' hypothesis.

2.1) In fact and for instance, the identification of the linguistic *and* with the lattice's conjunction ($\cdot$), presupposes that to have $x \cdot y$ given x and y, it should be known:

a) all the elements that are below x in the order $\leqslant$;
b) all the elements that are below y in the order $\leqslant$;
c) the intersection of these two sets of elements, and
d) that $x \cdot y$ is, exactly, the supremum of such intersection.

All that implies the need of a lot of information necessary for identifying 'x and y' with $x \cdot y$; an information not always actually available.

2.2) In addition to (2.1), in lattices the lattice conjunction is always commutative, $x \cdot y = y \cdot x$. But in Natural Languages, where almost always time intervenes and forces non-symmetry, commutativity is not necessarily always the case since often '*x and y'* has not the same meaning than '*y and x'*. It happens, for instance, with x = 'She entered into the room', and y = 'She started to cry'.

4 On the possibilistic interpretation of 'uncertain'

Mainly in the restricted setting of the De Morgan algebra of all fuzzy sets μ: $X \longrightarrow [0,1]$, given by the triplet (*min*, *max*, *1-id*), the predicate U is sometimes interpreted by the predicate $\boldsymbol{P}$ = possible [2]. This interpretation

also holds for the Boolean sub-algebra of crisp sets and, *mutatis mutandis* it also holds in any De Morgan algebra X, and in the sub-algebra of its Boolean elements [12].

Of course, the supposition that the universe of discourse where U is used is a De Morgan algebra also seems to be a hard one. Additionally, and for instance, in the case with fuzzy sets, and provided it is $\mu \cdot \mu' = \mu_0$ (law of non-contradiction), it is well known that no De Morgan algebra can structure the set of all fuzzy sets.

What can be only stated in general is that 'x is $\boldsymbol{P}$' => 'x is U', and that it can be supposed that $\leqslant_P \subseteq \leqslant_U$. Thus, if U is identified with P, then in the actual elemental meaning of U it will lack the difference-set $\leqslant_U - \leqslant_P$, provided it is non empty. Like it happens in the former case with P = probable, fuzzy logicians also seem to identify $\leqslant_P$ with the partial order $\leqslant$ of the De Morgan algebra, and try to measure the 'extent' up to which x is $\boldsymbol{P}$ by means of the so-called *measures of possibility*, mappings $Poss: X \longrightarrow [0,1]$ such that, provided X is a De Morgan (or a Boolean) algebra, verify:

1) $Poss(0) = 0$, 2) $Poss(1) = 1$, and 3) $Poss(x+y) = max(Poss(x), Poss(y))$, for all x, y in X,

from which it easily follows '$x \leqslant y$ => $Poss(x) \leqslant Poss(y)$', showing that *Poss* is a measure for $\boldsymbol{P}$ in the sense of section 1, and provided the preorder $\leqslant_P$ is equated with the partial order of the lattice.

Under this view, the measure of the statements 'x is U', once they are understood by 'x is $\boldsymbol{P}$', is done through a weak sub-additive condition than that in the probabilistic (additive) case, since it is $Poss(x+y) = max(Poss(x), Poss(y)) \leqslant Poss(x) + Poss(y)$, even if $x \cdot y = 0$. This 'possibilistic view' seems to be less restrictive than the probabilistic one. For instance, if X is a Boolean algebra endowed with a probability p 'consistent' with *Poss* [1], that is such that $p \leqslant Poss$, then *Poss* is sub-additive and larger than the additive p.

Remark Of course, although possibility measures are less informative than probability measures, also less previous data are necessary to define a measure *Poss*. For instance, in throwing an 'ideal' dice, it can be immediately said that *Poss* (obtaining 6) = 1, but p (obtaining 6) = 1/6. Nevertheless, in

the case of a dice loaded in such a way that three 6 will be reached at any 10 times of throwing it, is p (obtaining 6) = 0.3, but it could be taken *Poss* (obtaining 6)> 0.5, if it is only known that the dice is loaded with the aim of obtaining more times 6 than the other five points, an imprecise information with which no a single p can be defined, since the only that can be actually stated is:

$$p \text{ (obtaining 1)} + p \text{ (obtaining 2)} + \ldots + p \text{ (obtaining 5)} \leqslant 5/6,$$
$$\text{and } 1/6 \leqslant p \text{ (obtaining 6)}.$$

5 Comments

The linguistic term *uncertainty* is not only imprecise, but has a very broad applicative spectrum in both ordinary life and Science. To scientifically approach what it could mean, it seems necessary to previously capture how their, also imprecise, mother-predicate U = *uncertain* , the opposite of certain, is used in different contexts and with different purposes, and where can consequently be represented by a fuzzy set.

To such a goal, the membership function of U in the corresponding universe of discourse X should be carefully *designed*. That is, what should be designed is an approximation μ to the before introduced function m_U. This design can be done, as it is typical with fuzzy sets, once the available information on the behavior of U in X is, for instance, compacted in a set of rules (some of which will be necessarily imprecise) capturing, at least in part, the preorder $\leqslant_U$.

Since neither is always $\leqslant_U$ a total or linear preorder, nor it enjoys anti-symmetry, nor U is a crisp predicate, it should be noticed that its designed membership function μ almost always will be of an approximate character that, coming from the imperfections usually inherent to any process of design, impedes to confuse μ with m_U. In the usual case U is imprecise, m_U will rarely describe perfectly its use in X, that is, it rarely will be $\leqslant_U = \leqslant_{m_U}$, and less again $\leqslant_U = \leqslant_\mu$. Anyway, sometimes and in limiting cases, the use of U is crisp, that is, the universe X in which U is used, is partitioned in the two subsets respectively constituted by those x in X for which there is no doubt that x is fully U, and those for which is no doubt that x is fully not-U. In these cases the equivalence $=_U$ only gives two classes of equivalence, and the measure m_U not only coincides with μ, but degenerates to only taking the two extreme values 0 and 1, with m_U^{-1} (1) equal to one of these two classes in X, a classical subset perfectly describing U in X.

Hence, the usual imprecision of *U* is the main reason for which, in general, *it is impossible to be certain on uncertainty*. Under this view, it seems illusory to think on the possibility of an 'exact' study of uncertainty like it is, for instance, the algebraic study of finite dimensional real vector spaces, and it seems more liable to think on a kind of study similar to that of Thermodynamics and involving experimentation and theory. Such study will, nevertheless, require of some previous experimentation in specific contexts.

5.1

The idea behind the term 'uncertainty' is manifold, as it is, for instance, the case with 'vagueness'. This is clear enough after some usual utterances like 'Language is vague', and 'Life is uncertain', that show a doubtful generality for someone with a scientific-like orientation. For a direct study of 'uncertainty', it should be remarked that this term suffers a lack of concretion coming from that it is not a predicate, since there is no any known element *x* for which the statement '*x* is uncertainty' can have sense.
For instance, some scientifically-oriented people prefer to consider the term 'Fuzziness' than the term 'Vagueness', after the first was submitted to numerical measurement through the so-called sharpened order translating 'less fuzzy than' [13], and even if the first can be viewed as less general than the second. In the same vein, *uncertainty* seems, without a big risk of falling in just a chatting far from scientific methodology, to be in itself more the subject for a direct quantitative study than a topic to be looked at with only the glasses of logic, or probability, or possibility, that are surrounded with a lot of intellectual charge. Nevertheless, these approaches are, at least and in principle, good enough before starting with a more general reflection.

6 Conclusions

This paper briefly tries to pose a way to begin with the study of 'uncertainty' through its mother-predicate *U* = uncertain that, in front of the too general and abstract concept of uncertainty, has the advantage of being applicable to elements in several, and previously algebraically structured or not, universes of discourse and that, being an imprecise predicate, could be represented by using the machinery of fuzzy sets. For this goal, it is employed the typically scientific strategy of looking at 'uncertain' as a quantity given by the binary relation $\leqslant_U$ translating the empirical binary predicate 'less *U* than', and a numerical function or 'measure' m_U, that graduates the use of *U* in the corresponding universe *X*, by growing in the

unit interval with the 'growing under $\leqslant_U$, of the elements in X. In this way, U can be looked at as an 'ideal' fuzzy set in X. All that is not ideologically far from what is done in [14] with the term 'fuzziness' and once the mother-predicate 'fuzzy' is interpreted as a quantity making the subject 'fuzziness' more suitable than the philosophical term 'vagueness' for a scientific-like analysis difficult to do without previously capturing what it can mean 'less vague than'. The predicate 'uncertain', like many other concepts in science, does not enjoy the possibility of being characterized by an if-and-only-if definition, something that (limits, if not) impedes a strictly formal kind of research on uncertainty, and supports its consideration as a fuzzy set.

6.1

There are two perhaps interesting facets in the current approach. The first is that the universe X does not necessarily require to be previously endowed with an algebraic structure, able for some mathematical developments, but not directly related to the use of U in it. The second is the drawback meant by how difficult is usually to completely know both the empirically-based relation $\leqslant_U$, and the measure m_U. The first allows to distinguish between the requirements of U and those of the mathematical tools employed for its study. The second is common for most fuzzy sets coming from non-crisp linguistic labels, and is relatively softened by 'designing' an approximation to m_U with the best information directly available on the behavior of U in X, something depending on how U can be understood through their contextual and purpose-driven use. Background, and contextual, knowledge are essential for a correct design of a fuzzy set labeled U.

For instance, with an uncertain statement of random-type like 'x is five', where x represents a result obtainable after throwing a dice, and under the supposition that the throwing can be, at least ideally, repeated an infinite number of times, it is possible to accept that the set of statements 'x is A', with A an statement on such result, is an atomic Boolean algebra. This supposition allows to interpret, for instance and with A = odd, 'x is A' as " 'x is 1' or 'x is 3' or 'x is 5' ", or with A = 'neither 3, nor 6', as coincidental with " 'x is 1' or 'x is 3' or 'x is 4' or 'x is 5' ", and so on. In cases like these, in which it is well imaginable that the 'experiment' can be repeated a lot of times, the uncertainty of these statements can be measured by a Kolmogorov's probability p, defined in such Boolean algebra by p('x is n') =a_n ($1\leqslant n\leqslant 6$), in such a way that it is not only $0 \leqslant a_n \leqslant 1$, but $a_1 + a_2 + ...+ a_6 = 1$. Of course, if the throwing of the dice is ideal or perfect, it is $a_1= a_2= ...=a_6= 1/6$, and any

other sequence (a_1,...,a_6) different from (1/6,...,1/6) indicates that the dice is either imperfectly made, or tricky (for instance, loaded).
Notice that if, for instance, *A* and *B* are the events respectively consisting in obtaining, respectively, '1 or 2', and '3 or 4', it is not *A* included in *B*, but $p(A) = p(B)$, showing the incomparability between $\leqslant$ and $\leqslant_p$.
In elemental examples like this last, $p(x \text{ is } A)$ can well represent a measure of the uncertainty the statement '(x is A) is probable' is subjected to, and that, without too many doubts, can approach the measure of the statement '(x is A) is uncertain' although there is no coincidence between the three relations $\leqslant$, $\leqslant_U$, and $\leqslant_P$.

6.2.

Random uncertainty is not the only type of uncertainty appearing in the world, or, at least in social environments. For instance, one can be reasonably uncertain on statements like 'Jean is neither young, nor old', 'John's apartment is big', 'Michael's Bank Account stores a lot of money', that cannot be submitted to any imaginable kind of repetition and depend on the corresponding contexts and on what it is understood by young, old, big, a lot. If the uncertainty of the random (repetitive) events can be measured by means of Kolmogorov's probabilities, which kind of measures m_U can be useful for the statements not susceptible to repetition?
Even for the case of fuzzy but not crisp events, an example of this measures with a compact set $X \subseteq R$, is shown by the possibility measures in section 3, whose values can be usually obtained (if the available information on the statement *A* is numerically done by a function $I: X \longrightarrow [0,1]$ such that $Sup\ I = 1$) by the formula,

$$Poss\ (A) = Sup\ \{\ x \text{ in } X;\ min\ (m_A\ (x), I(x))\}\ ,$$

that, when *X* is finite gives the only way for obtaining all the '*Poss*' measures. This formula was introduced in [15], either holds if *A* is a precise or an imprecise statement, and depends on the available information on the statement *A*, when such information can be represented by a function *I*. An example is given, for instance, when on a variable *Z* varying in the unit interval, it is only known that *Z* does not take any value below *1/4* but takes values above *3/4*, and it is asked how uncertain can be the statement '*Z* is not small'. In this case, the available although incomplete information on the behavior of *Z*, can be given by the function $I: [0,1] \longrightarrow [0,1]$, defined by

$I(x) = 0$, if $0 \leqslant x \leqslant 1/4$;
$I(x) = 1$, if $3/4 < x \leqslant 1$, and
$I(x) = 2x - 0.5$, if $1/4 \leqslant x \leqslant 3/4$,

and by 'small' represented by the function $m_A(x) = 1-x$. Once all that is placed in the above formula, the obtained result $Poss(A) = 0.5$, shows the amount of uncertainty associated with A under the interpretation of U by $\boldsymbol{P}$.

6.3

Basic for the study of what 'uncertain' means, are the restrictions imposed on the graph $(X, \leqslant_U)$ when more properties coming from the particular use of U on X can be added. In this sense, for instance, a better understanding of the actual possibility of identifying the preorder $\leqslant_U$ with an structural relation, or a partial order, in the universe of discourse and in agreement with the use of U, as it is supposedly done with the predicates P and $\boldsymbol{P}$, is of a paramount importance. This is the case of the probability's identification of $\leqslant_P$ with the 'inclusion' of events in the case of random-type uncertainty, that is, to know if each time the random-event A is 'included' in the random-event B ($A \subseteq B$), it can be equivalently stated that A is 'less probable than' B. In the same vein, it is important to better know if each time it is $A \leqslant_U B$, it is or it is not $A \leqslant_P B$, and the same question is relevant when instead of P it is considered $\boldsymbol{P}$.
To create a true new field of research for a Science of Uncertainty, it yet lacks a lot of intensive research's efforts. Nevertheless, such new field can be of a great relevance, and this paper only should be viewed 'cum grano salis' in the path to such scientific endeavour.

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