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Fuzzy in 3–D: Two Contrasting Paradigms

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Abstract Type-2 fuzzy sets and complex fuzzy sets are both three dimensional extensions of type-1 fuzzy sets.

Complex fuzzy sets come in two forms, the standard form, postulated in 2002 by Ramot et al., and the 2011 innovation of pure complex fuzzy sets, proposed by Tamir et al.. In this paper we compare and contrast both forms of complex fuzzy set with type-2 fuzzy sets, as regards their rationales, applications, definitions, and structures. In addition, pure complex fuzzy sets are compared with type-2 fuzzy sets in relation to their inferencing operations.

Complex fuzzy sets and type-2 fuzzy sets differ in their roles and applications; complex fuzzy sets are pertinent to inferencing where there is seasonality, and type-2 fuzzy sets are applicable to reasoning under uncertainty. Their definitions differ also, though there is equivalence between those of a pure complex fuzzy set and a type-2 fuzzy set. Structural similarity is evident between these three-dimensional sets. Complex fuzzy sets are represented by a 3–D line, and type-2 fuzzy sets by a 3–D surface, but a surface is simply a generalisation of a line. This similarity is particularly apparent between pure complex fuzzy sets and type-2 fuzzy sets, which are both mappings from the domain onto the unit square. However type-2 fuzzy sets were found not to be isomorphic to pure complex fuzzy sets.

The mechanisms by which complex fuzzy sets model and quantify periodicity, and type-2 fuzzy sets model and quantify uncertainty are discussed.

A type-2 fuzzy set can be represented as the union of its type-2 embedded set. An embedded type-2 fuzzy set is a type-2 fuzzy set in itself, whose geometrical representation is a 3-D line. Thus, geometrically an embedded type-2 fuzzy set can be seen as equivalent to a pure complex fuzzy set and therefore a type-2 fuzzy set can be represented as the union of a collection pure complex fuzzy sets, which in turn can be regarded as embedded complex fuzzy sets of a type-2 fuzzy set. This relationship is exploited to provide a complex definition of a type-2 fuzzy set.

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1 Introduction

It is 50 years since Zadeh [1] introduced the concept of a type-1 fuzzy set. In 1975, he went on to define the *type-2 fuzzy set* [2–4]. *Complex fuzzy sets*, applicable to the modelling of periodic phenomena, are a relatively recent development in fuzzy set theory, proposed by Ramot et al. in 2002 [5]. A natural extension of real-valued, or type-1 fuzzy sets, they differ from them in so far as their membership grades are complex numbers (of modulus ≤ 1) [6]. A striking feature that type-2 fuzzy sets and complex fuzzy sets have in common is that they are three-dimensional.

Pure complex fuzzy sets are a very recent variation on complex fuzzy sets proposed by Tamir et al. in 2011. Using Cartesian coordinates, both the real and imaginary components of the membership grade may take any value within the interval $[0, 1]$ [7, page 293]. Pure complex fuzzy sets may also be represented using polar coordinates [7, page 294] in a formalisation that is superficially similar to that of complex fuzzy sets as defined by Ramot et al., but in which the phase and modulus terms are interpreted differently.

In this paper, in order to distinguish between pure complex fuzzy sets and the original complex fuzzy sets proposed by Ramot et al., we shall refer to complex fuzzy sets as first postulated as *standard complex fuzzy sets*. The phrase 'complex fuzzy set' will refer to either form.

Type-2 fuzzy sets [2–4] are an extension of type-1 fuzzy sets in which the sets' membership grades are themselves type-1 fuzzy sets. They respond to a major shortcoming of type-1 fuzzy sets by offering a conceptual scheme within which the effects of uncertainties in fuzzy inferencing may be modelled and minimised [8, page 117].

The purpose of this paper is to establish similarities and differences firstly between complex fuzzy sets and type-2 fuzzy sets, and secondly between complex fuzzy inferencing systems and type-2 fuzzy inferencing systems. The report is structured as follows: After the introduction of this section, Section 2 sets out the definitions of the fuzzy sets. In Section 3, the structures of the sets are discussed, and in Section 4, the inferencing operations are investigated. A definition of type-2 fuzzy sets in terms of complex fuzzy sets is presented in Section 5. The final section, Section 6, concludes the paper.

1.1 Fuzzy Inferencing Systems

It is via the Fuzzy Inferencing System (FIS)[1] that fuzzy sets are put to use. An FIS is a decision making program which works by applying fuzzy logic operators to common-sense linguistic rules. In this paper we are concerned with the Mamdani FIS, in which a crisp numerical input passes through three stages: fuzzification, inferencing, and finally defuzzification. The output of inferenc-

[1] A Fuzzy Inferencing System is also known as a *Fuzzy Logic System (FLS)*.

ing is a fuzzy set known as the *aggregated set*. During the defuzzification stage the aggregated set is converted into a crisp number, which is the output of the FIS. Figure 1 provides a representation of a Mamdani-style type-2 FIS. A Mamdani-style complex FIS differs from the type-2 version only in that defuzzification is a one stage procedure. Defuzzification is beyond the scope of this paper; interested readers are referred to the articles of Greenfield et al. [9–12]. The focus of this present paper will be restricted to the operations in the inferencing stages of the FIS (Section 4).

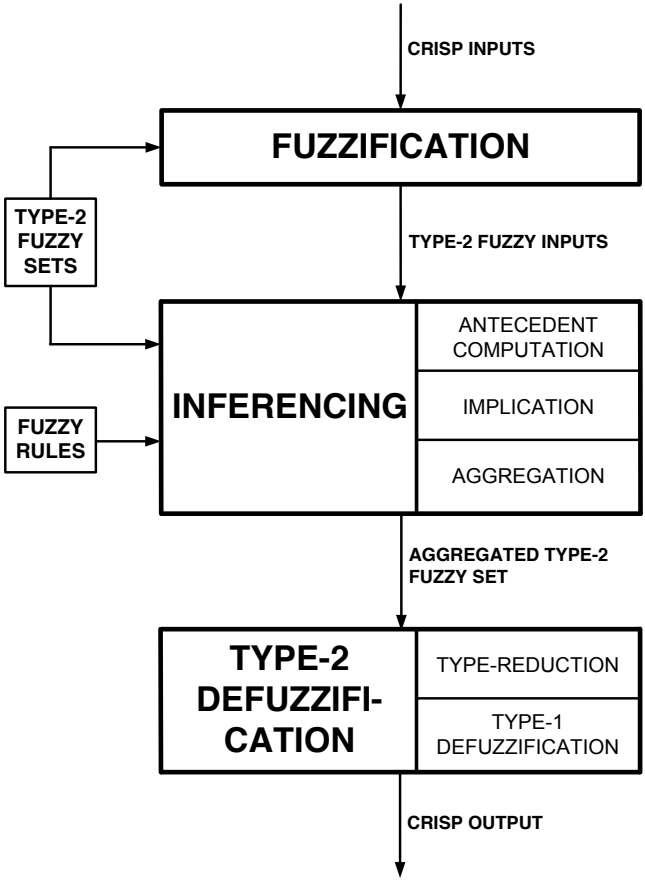


Figure 1: Mamdani-style Type-2 FIS.

1.2 Complex Fuzzy Sets

Ramot et al. [5, page 171] present standard complex fuzzy sets as an extension of type-1 fuzzy sets:

> "... complex fuzzy set theory modifies the original concept of fuzzy membership by asserting that, at least in some instances, it is necessary to add a second dimensions to the expression of membership. ...The novelty of complex fuzzy sets is manifested in the additional dimension of membership: the *phase* of the grade of membership, $\omega_S(x)$. "

And what is the purpose of this additional phase term? It permits complex fuzzy sets to intuitively represent fuzziness in time series applications. Dick gives this example [6, page 413]:

> "... consider traffic congestion in a major city. The basic dynamics of traffic congestion are depressingly well-known: there is a morning "rush hour" when workers are trying to get to their jobs, causing high congestion in one direction on major roads; in the afternoon, there is a rush hour in the opposite direction, as everyone goes home. In between, traffic is lighter, and at night, the roads are nearly empty. This situation is *approximately* periodic, but never *exactly* repeats itself. Zadeh has recently [13] termed this phenomenon "regularity," ..."

Dick goes on to say that in his view,

> "... the proper role for a complex fuzzy set is a remarkably efficient representation of approximately periodic phenomena, and as the underlying mathematical foundation of regularity."

Applications of complex fuzzy sets include analysing solar activity as measured by the recorded number of sunspots [5], signal processing [5], stock trading on the New York Stock Exchange [14], and prediction of voter turnout in elections [14]. The potential for further applications is enormous.

1.3 Type-2 Fuzzy Sets

Type-1 membership functions are subject to uncertainty arising from various sources [8]. Their accuracy is therefore questionable; it seems counterintuitive to use real numbers, possibly expressed to several decimal places, to represent degrees of membership. Klir and Folger comment [15, page 12]:

> "... it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real numbers. Although this does not pose a

> serious problem for many applications, it is nevertheless possible to extend the concept of the fuzzy set to allow the distinction between grades of membership to become blurred. Sets described in this way are known as *type 2 fuzzy sets*."

Here Klir and Folger describe blurring a type-1 fuzzy set to form an *interval type-2 fuzzy set*. Mendel and John take this idea a stage further [8, page 118], describing the transition from a type-1 fuzzy set to a *generalised type-2 fuzzy set*, again by blurring the type-1 membership function:

> Imagine blurring the type-1 membership function [...] by shifting the points [...] either to the left or the right, and not necessarily by the same amounts, [...]. Then, at a specific value of x, say x', there no longer is a single value for the membership function (u'); instead the membership function takes on values wherever the vertical line [$x = x'$] intersects the blur. These values need not all be weighted the same; hence, we can assign an amplitude distribution to all of these points. Doing this for all $x \in X$, we create a three-dimensional membership function — a type-2 membership function — that characterizes a type-2 fuzzy set.

So, type-2 fuzzy sets have a third dimension. This has advantages and disadvantages. From a modelling perspective type-2 sets provide more *degrees of freedom*.

Thus type-2 fuzzy sets take two forms, the interval, for which all secondary membership grades are 1, and the generalised, where the secondary membership grade may take any value between 0 and 1. The tendency has been for developers to opt [16, pages 7, 8, 16] for the computationally simpler interval type-2 FISs [8, 17] for which applications have been developed in areas such as control, simulation and optimisation [18–23]. In contrast, there are relatively few, though varied, generalised type-2 fuzzy applications [8, 24, 25]. Since strategies have been and continue to be developed that reduce the computational complexity of all stages of the generalised type-2 FIS [10,12,26–28], it is to be hoped that in the future there will be an increasing number of generalised type-2 FIS applications.

For a complex fuzzy set, the third dimension conveys additional information, namely phase, which intuitively models and quantifies the stage in the periodic cycle. In contrast, for a type-2 fuzzy set, the third dimension reflects the uncertainty arising out of a deficit in information. How type-2 fuzzy sets model uncertainty is less obvious and is the subject of [29]. In this paper it is argued that the volume under the surface of the type-2 fuzzy set is a measure of the uncertainty relating to the set.

2 Definitions

2.1 Type-1 Fuzzy Sets

Since complex fuzzy sets and type-2 fuzzy sets are both extensions of the basic type-1 fuzzy set, we begin by formally defining the type-1 fuzzy set.

Definition 1 (Type-1 Fuzzy Set)**.** *Let X be a universe of discourse. A fuzzy set A in X is characterised by a membership function $\mu_A : X \to [0, 1]$, and can be expressed as follows:*

$$A = \{(x, \mu_A(x)); \mu_A(x) \in [0, 1] \, \forall x \in X\}. \tag{1}$$

Note that the membership grades of A are crisp, real numbers.

2.2 Complex Fuzzy Sets

Standard complex fuzzy sets are defined using polar co-ordinates.

Definition 2 (Standard Complex Fuzzy Set [5, page 172])**.** *"A complex fuzzy set S, defined on a universe of discourse U, is characterized by a membership function $\mu_S(x)$ that assigns any element $x \in U$ a complex-valued grade of membership in S. By definition, the values $\mu_S(x)$ may receive all lie within the unit circle in the complex plane, and are thus of the form $r_S(x) \cdot e^{j\omega_S(x)}$, where $j = \sqrt{-1}$, $r_S(x)$ and $\omega_S(x)$ are both real-valued, and $r_s(x) \in [0, 1]$.*

The complex fuzzy set S may be represented as the set of ordered pairs

$$S = (x, \mu_S(x))|x \in U."$$

Though this definition employs polar coordinates, conversion between the polar form and the Cartesian form is straightforward; in Figure 4(b) the data displayed is in Cartesian form.

The Cartesian representation of a pure complex fuzzy grade of membership

$$\mu(V, z) = \mu_r(V) + j\mu_i(z) \tag{2}$$

is defined by Tamir et al. [7] thus:

- $\mu_r(V)$ is the grade of membership in the interval $[0, 1]$ of a set V in a fuzzy class Γ (a fuzzy set of fuzzy sets), i.e. is a 'pointer' to a (fuzzy) set V within a fuzzy class, and
- $\mu_i(z)$ is the grade of membership in the interval $[0, 1]$ of the element z in the fuzzy set V, i.e. 'pointer' to an element within the fuzzy set V.

This leads to a formal definition of a pure complex fuzzy set:

Definition 3 (Pure Complex Fuzzy Set [7])**.** *Let $\tilde{P}(T)$ be the set of fuzzy sets in T and $\Gamma \subseteq \tilde{P}(T)$. A pure complex fuzzy set $\tilde{C}$ on T is characterised by a pure complex membership function*

$$\mu \colon \Gamma \times T \longrightarrow \mathbb{C}$$

$$\mu(V, z) = \mu_r(V) + j\mu_i(z)$$

where $\mu_r(V)$ is the grade of membership in the interval $[0, 1]$ of a set V in the fuzzy class Γ and $\mu_i(z)$ is the grade of membership in the interval $[0, 1]$ of the element z in the fuzzy set V.

2.3 Type-2 Fuzzy Sets

Let X be a universe of discourse. Let $\tilde{P}(X)$ be the set of fuzzy sets in X. A type-2 fuzzy set $\tilde{A}$ in X is a fuzzy set whose membership grades are themselves fuzzy. This implies that $\mu_{\tilde{A}}(x)$ is a fuzzy set in $[0, 1]$ for all x, i.e.

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); \mu_{\tilde{A}}(x) \in \tilde{P}([0, 1]) \, \forall x \in X\}. \tag{3}$$

It follows that $\forall x \in X \; \exists J_x \subseteq [0, 1]$ such that $\mu_{\tilde{A}}(x) : J_x \to [0, 1]$. Applying (1), we have:

$$\mu_A(x) = \{(u, \mu_{\tilde{A}}(x)(u)); \mu_{\tilde{A}}(x)(u) \in [0, 1] \, \forall u \in J_x \subseteq [0, 1]\}. \tag{4}$$

J_x is called the primary membership of x while $\mu_{\tilde{A}}(x)$ is called the secondary membership of x.

Putting (3) and (4) together we have

Definition 4 (Type-2 Fuzzy Set)**.**

$$\tilde{A} = \{(x, (u, \mu_{\tilde{A}}(x)(u)))|\mu_{\tilde{A}}(x)(u) \in [0, 1] \forall x \in X \wedge \forall u \in J_x \subseteq [0, 1]\}, \tag{5}$$

where X is a universe of discourse and $\tilde{A}$ is a type-2 fuzzy set in X.

Two concepts relating to type-2 fuzzy sets are the footprint of uncertainty and the vertical slice.

Definition 5 (Footprint Of Uncertainty)**.** *The* Footprint Of Uncertainty (FOU) *is the projection of the type-2 fuzzy set onto the $x - u$ plane.*

Definition 6 (Vertical Slice)**.** *A* vertical slice *is a plane which intersects the x-axis (primary domain) and is parallel to the u-axis (secondary domain).*

The notion of a vertical slice may be extended to complex fuzzy sets in both the standard and pure forms.

3 Structure

Figure 2 shows a type-2 fuzzy set (from a Matlab^{TM} application), together with its FOU. Figure 3 shows the conventional 2-D representation of the time series consisting of sunspot numbers observed on a monthly basis [30]. Figure 4 shows the sunspot observations of Figure 3 [30] displayed as a complex fuzzy set.

A standard complex fuzzy set is represented mathematically by a mapping whose range is the unit disc, centre $(0, 0)$. In contrast, the mapping representing a pure complex fuzzy set has the unit square, with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$, as its range. Similarly, the range of the type-2 fuzzy set mapping is the unit square.

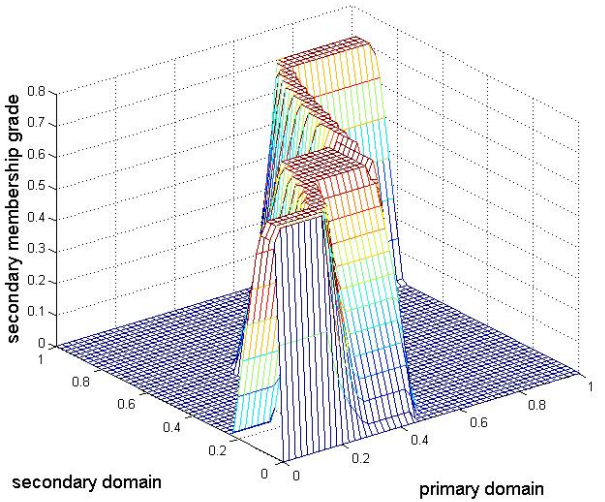
What is the difference in structure between a complex fuzzy set and a generalised type-2 fuzzy set? Might not a complex fuzzy set be seen a special case of a generalised type-2 fuzzy set? Graphically a type-2 fuzzy set is a surface in 3-D, whereas a complex fuzzy set is a line in 3-D. But a line is a specific sort of surface. So structurally a complex fuzzy set may be considered to be a special form of generalised type-2 fuzzy set (in the same way that an interval type-2 fuzzy set is a specialisation of the generalised type-2 fuzzy set). This is discussed further in Section 5.

4 Operations

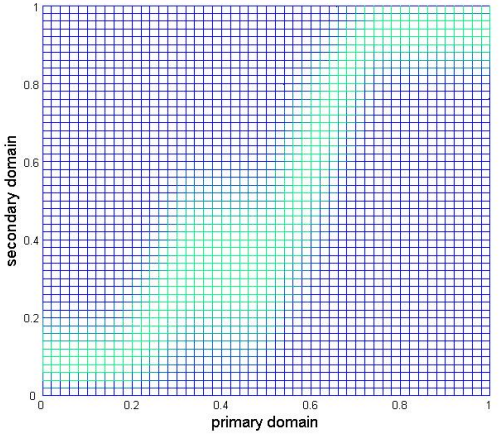
Within a type-2 FIS, join and meet operations are used extensively. Similarly, for a complex FIS, union and intersection operations are pervasive. For both type-2 and complex FISs, all the computation occurring during the major stage of inferencing is founded upon these operations. In this section we will look at whether the type-2 fuzzy join and meet operations are transferable to complex fuzzy sets.

To facilitate comparison between mappings, it is essential that they take the same range, otherwise a bijective transformation between the ranges would need to be exhibited. As this is not a trivial issue, we restrict our analysis to comparing type-2 fuzzy sets with pure complex fuzzy sets.

Minimum ('$\wedge$' or '$\odot$') is the most frequently used t-norm; in the analysis which follows, this t-norm is employed, as is the maximum t-conorm ('$\vee$' or '$\oplus$').



(a) 3-D representation



(b) FOU

Figure 2: Aggregated type-2 fuzzy set created during the inference stage of a type-2 FIS.

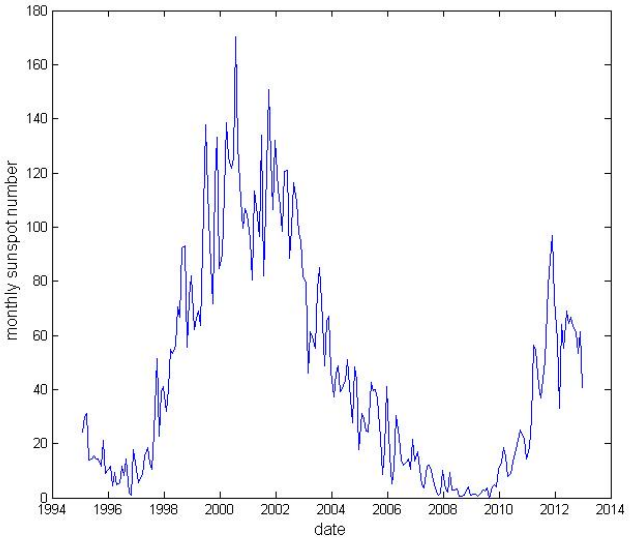
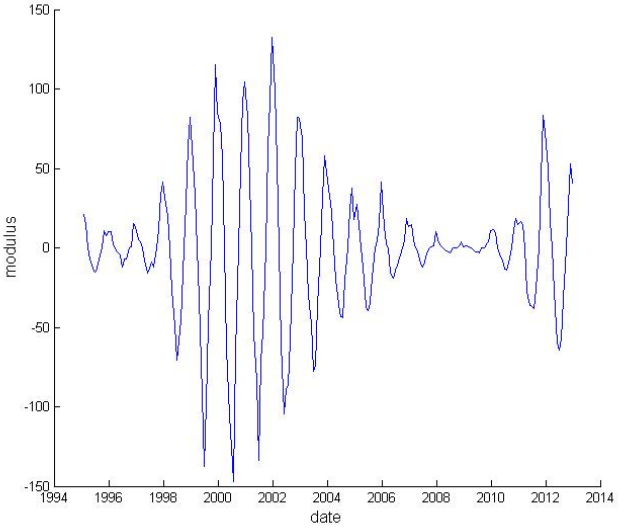
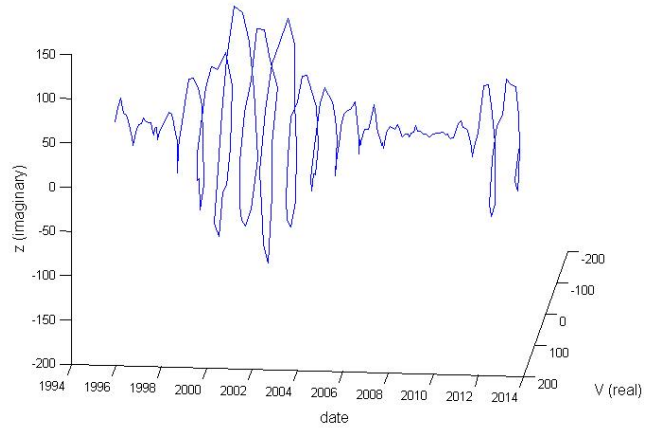


Figure 3: Number of sunspots recorded on a monthly basis between 1994 and 2013.



(a) 2-D view.



(b) 3-D view.

Figure 4: Sunspot data represented as a complex fuzzy set. The modulus is greater than 1 because the sunspot data has not been normalised.

4.1 Operations on Pure Complex Fuzzy Sets

4.1.1 Equation for Union

According to Tamir et al. [7, pages 299–300] there are three ways to construct the union of two pure complex fuzzy classes[2]. They describe this (their third) construction as "…sound, intuitive, and practical."

Definition 7 (Union of Pure Complex Fuzzy Classes)**.** *Let* $\Gamma = \{V, z, \mu_\Gamma(V, z) | V \in 2^U, z \in U\}$ *and* $\Psi = \{T, z, \mu_\Psi(T, z) | T \in 2^U, z \in U\}$ *be two complex fuzzy classes such that* V *and* T *are fuzzy sets. Assume that* Γ *and* Ψ *are defined over a universe of discourse* U*, and let* 2^U *denote the power set of* U*. Further assume that the degree of membership of an object* $z \in V$ *and an object* $y \in T$ *is given by* $\mu_\Gamma(V, z) = \mu_{\Gamma_r}(V) + j\mu_{\Gamma_i}(z)$ *and* $\mu_\Psi(T, y) = \mu_{\Psi_r}(T) + j\mu_{\Psi_i}(y)$*, respectively, where* $\mu_{\Gamma_r}(\alpha)$*,* $\mu_{\Psi_r}(\alpha)$*,* $\mu_{\Gamma_i}(\alpha)$*, and* $\mu_{\Psi_i}(\alpha)$ *stand for the real and imaginary parts of* $\mu_\Gamma(V, x)$ *and* $\mu_\Psi(T, y)$*. Finally, let* $W = 2^U$*, and let* $\oplus$ *denote a t-conorm operation. Then*

$$\mu_{\Gamma \cup \Psi}(W, z) = (\mu_{\Gamma_r}(V) \oplus \mu_{\Psi_r}(T)) + j(\mu_{\Gamma_i}(z) \oplus \mu_{\Psi_i}(z)). \tag{6}$$

4.1.2 Equation for Intersection

The position with respect to intersection is analogous to that of union, and again the definition presented here is described by Tamir et al. as "…sound, intuitive, and practical" [7, pages 301–302].

Definition 8 (Intersection of Pure Complex Fuzzy Classes)**.** *Let* $\Gamma = \{V, z, \mu_\Gamma(V, z) | V \in 2^U, z \in U\}$ *and* $\Psi = \{T, z, \mu_\Psi(T, z) | T \in 2^U, z \in U\}$ *be two complex fuzzy classes such that* V *and* T *are fuzzy sets. Assume that* Γ *and* Ψ *are defined over a universe of discourse* U*, and let* 2^U *denote the power set of* U*. Further assume that the degree of membership of an object* $z \in V$ *and an object* $y \in T$ *is given by* $\mu_\Gamma(V, z) = \mu_{\Gamma_r}(V) + j\mu_{\Gamma_i}(z)$ *and* $\mu_\Psi(T, y) = \mu_{\Psi_r}(T) + j\mu_{\Psi_i}(y)$*, respectively, where* $\mu_{\Gamma_r}(\alpha)$*,* $\mu_{\Psi_r}(\alpha)$*,* $\mu_{\Gamma_i}(\alpha)$*, and* $\mu_{\Psi_i}(\alpha)$ *stand for the real and imaginary parts of* $\mu_\Gamma(V, x)$ *and* $\mu_\Psi(T, y)$*. Finally, let* $W = 2^U$*, and let* $\odot$ *denote a t-norm operation. Then*

$$\mu_{\Gamma \cap \Psi}(W, z) = (\mu_{\Gamma_r}(V) \odot \mu_{\Psi_r}(T)) + j(\mu_{\Gamma_i}(z) \odot \mu_{\Psi_i}(z)). \tag{7}$$

4.1.3 Union and Intersection Performed Slice by Slice

Union and intersection operations proceed slice by slice, so it is sufficient to specify how these operations may be applied to two slices. For union, Equation 6 requires that the t-conorm operator be applied to both the real and imaginary components of the membership grade i.e the $\max / \max$ combination.

[2] Tamir et al. present their analysis in terms of *classes*, but for the purposes of this paper it can be assumed that classes and sets are equivalent.

For intersection, Equation 7 indicates that the t-norm operator is applicable to both the real and imaginary components of the membership grade i.e the $\min / \min$ combination.

4.2 Operations on Type-2 Fuzzy Sets

4.2.1 Equation for Join

The formula for the join operation is:

$$\mu_{\tilde{A} \cup \tilde{B}}(x) = \int_{u \in J_x^u} \int_{w \in J_x^w} f_x(u) \star g_x(w) / (u \vee w) \quad x \in X,$$

where $\vee$ is the maximum operator, $\star$ signifies a t-norm, $\iint$ represents union over $J_x^u \times J_x^w$, and $f_x(u)$ and $g_x(w)$ are respectively the corresponding secondary grades of $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ [31, pages 217–18].

4.2.2 Equation for Meet

The formula for the meet operation is:

$$\mu_{\tilde{A} \cap \tilde{B}}(x) = \int_{u \in J_x^u} \int_{w \in J_x^w} f_x(u) \star g_x(w) / (u \wedge w) \quad x \in X,$$

where $\wedge$ is the minimum operator, $\star$ signifies a t-norm, $\iint$ represents union over $J_x^u \times J_x^w$, and $f_x(u)$ and $g_x(w)$ are respectively the corresponding secondary grades of $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ [31, page 219].

4.2.3 Join and Meet Performed Slice by Slice

Join and meet operations proceed slice by slice, so it is sufficient to specify how these operations may be applied to two slices.

Let $\tilde{A}$ and $\tilde{B}$ be two type-2 fuzzy sets, in which the co-domains are discretised into N slices, and the domains sliced at the points $x_{\tilde{A}}$ and $x_{\tilde{B}}$ respectively. Two type-1 fuzzy sets,

$$S_{\tilde{A}} = \{z_{A_1} / y_{A_1} + z_{A_2} / y_{A_2} + \cdots + z_{A_N} / y_{A_N}\}$$

and

$$S_{\tilde{B}} = \{z_{B_1} / y_{B_1} + z_{B_2} / y_{B_2} + \cdots + z_{B_N} / y_{B_N}\},$$

are generated.

Join The formula for join requires that all N^2 possible $\min / \max$ pairings of $S_{\tilde{A}}$ and $S_{\tilde{B}}$ be created:

$$\min(z_{A_1}, z_{B_1}) / \max(y_{A_1}, y_{B_1}) + \min(z_{A_1}, z_{B_2}) / \max(y_{A_1}, y_{B_2}) + \cdots$$
$$\cdots + \min(z_{A_N}, z_{B_N}) / \max(y_{A_N}, y_{B_N}).$$

Meet Similarly, for meet, pairings are generated as follows:

$$\min(z_{A_1}, z_{B_1})/\min(y_{A_1}, y_{B_1}) + \min(z_{A_1}, z_{B_2})/\min(y_{A_1}, y_{B_2}) + \cdots$$

$$+ \cdots \min(z_{A_N}, z_{B_N})/\min(y_{A_N}, y_{B_N}).$$

Selection of Maximum Membership Grade The next stage is the same for join and meet. For every resultant domain value ('denominator') generated, the maximum membership grade ('numerator') is selected. The resultant set of pairs is the join or meet of the two slices.

4.3 Applying Join and Meet of Slices to Complex Fuzzy Sets

What happens when the join and meet operations of type-2 fuzzy sets are substituted for the union and intersection operations of (pure) complex fuzzy sets? The complex fuzzy set would be regarded as a type-2 fuzzy set whereby for each vertical slice the co-domain is not discretised i.e. $N = 1$. It follows that $N^2 = 1$, indicating that there is only 1 $\min$ / $\max$ pairings of $S_{\tilde{A}}$ and $S_{\tilde{B}}$ in the case of join (union) and only 1 $\min$ / $\min$ pairings in the case of meet (intersection). There would be no need to select the maximum membership grade as there would only be one pair; this stage is superfluous. The resultant set of one pair is the join or meet of the two slices.

Is pure complex fuzzy inferencing isomorphic to type-2 fuzzy inferencing? The type-2 meet operation carries over to the pure complex intersection operation as $\min$ / $\min$. However the type-2 join operation is not transferrable to the pure complex union operation; the former is $\min$ / $\max$, whereas the latter is $\max$ / $\max$. So there is no isomorphism between pure complex fuzzy sets and type-2 fuzzy sets.

5 Complex Fuzzy Set definition of a Type-2 Fuzzy Set

The similarity between the concepts of type-2 fuzzy sets and complex fuzzy sets is apparent from the definition of Tamir et al. of a complex fuzzy class [7, page 295] as " …a pure fuzzy class of order one, that is, a fuzzy set of fuzzy sets." A complex fuzzy class, defined over a universe of discourse T, is characterised by a pure complex membership function $\mu(V, z)$ that assigns a complex-valued grade of membership in the complex fuzzy class to any element $z \in U$ (where U is the universe of discourse). The values that $\mu(V, z)$ may receive lie within the unit square or the unit circle in the complex plane

$$\mu(V, z) = \mu_r(V) + j\mu_i(z) \tag{8}$$

where $\mu_r(V)$ and $\mu_i(z)$ are real value fuzzy grades of membership in the interval $[0, 1]$. Therefore, Tamir et al. [7, page 297] consider "…the definition

of a pure fuzzy class (a class of order 1) as a mapping into a two-dimensional space."

Fuzzy sets of type-2, in which the membership grades are themselves fuzzy sets, are known as 'fuzzy-fuzzy sets', a description that resembles the definition above of the complex fuzzy class.

An embedded type-2 fuzzy set is a special kind of type-2 fuzzy set. It relates to the type-2 fuzzy set in which it is embedded in this way: For every primary domain value, x, there is a unique secondary domain value, u, plus the associated secondary membership grade that is determined by the primary and secondary domain values, $\mu_{\tilde{A}}(x)(u)$. For discrete universes of discourse X (or T) and U, we have the following mathematical expression of an embedded set:

Definition 9 (Embedded Set)**.** *Let $\tilde{A}$ be a type-2 fuzzy set in X. For discrete universes of discourse X and U,* an embedded type-2 set $\tilde{A}_e$ *of $\tilde{A}$ is defined as the following type-2 fuzzy set*

$$\tilde{A}_e = \{(x_i, (u_i, \mu_{\tilde{A}}(x_i)(u_i)))|\ \forall i \in \{1, \ldots, N\} :\ x_i \in X\ \ u_i \in J_{x_i} \subseteq U\}. \quad (9)$$

$\tilde{A}_e$ contains exactly one element from J_{x_1}, J_{x_2}, …, J_{x_N}, namely u_1, u_2, …, u_N, each with its associated secondary grade, namely $\mu_{\tilde{A}}(x_1)(u_1)$, $\mu_{\tilde{A}}(x_2)(u_2)$, …, $\mu_{\tilde{A}}(x_N)(u_N)$.

Tamir et al. interpreted the above Cartesian representation of a complex fuzzy class [7, page 293] such that the real part acts as a "pointer to a set within a fuzzy class" and the imaginary part a "pointer to an item within a fuzzy set." A similar interpretation can be associated with the values u_i and $\mu_{\tilde{A}}(x_i)(u_i)$ in the definition above of an embedded type-2 fuzzy set. The value u_i can be seen as a pointer to the primary membership set J_x, while the value $\mu_{\tilde{A}}(x_i)(u_i)$ points to the item within the set J_x. Thus, the embedded type-2 fuzzy set $\tilde{A}_e$ of type-2 fuzzy set $\tilde{A}$ can be regarded as mathematically equivalent to the complex fuzzy class with following complex-value grade of membership

$$\mu(V, z) = \mu_r(V) + j\mu_i(z)$$

where V is a subset of U, $z \in X$ and

$$\mu_r(V) = \begin{cases} u_i \in J_x & \text{if } V = J_x \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_i(z) = \begin{cases} \mu_{\tilde{A}}(x_i)(u_i) & \text{if } V = J_x \\ 0 & \text{otherwise.} \end{cases}$$

A type-2 fuzzy set can be represented as the union of its type-2 embedded sets. Let $\tilde{A}_e^j$ denote the jth type-2 embedded set for type-2 fuzzy set $\tilde{A}$, i.e.,

$$\tilde{A}_e^j \equiv \left\{ \left(u_i^j, \mu_{\tilde{A}}(x_i)(u_i^j) \right), i = 1, \ldots, N \right\}$$

where $\{u_i^j, \ldots, u_N^j\} \in J_{x_i}$. Then $\tilde{A}$ can be represented as the union of its type-2 embedded sets, i.e., $\tilde{A} = \sum_{j=1}^{n} \tilde{A}_e^j$ where $n \equiv \prod_{i=1}^{N} M_i$. Therefore, we can define a type-2 fuzzy set as a collection of embedded complex fuzzy classes in accord with the above mathematical representation.

6 Conclusions and Further Work

This paper has evaluated the similarities and distinctions between complex fuzzy sets and systems and type-2 fuzzy sets and systems in several respects:

Rationale For a complex fuzzy set, the third dimension reflects additional information — that of phase. However, for a type-2 fuzzy set, the third dimension reflects the uncertainty arising out of a deficit in information.

Applications Complex fuzzy sets are applicable to the analysis of time series, where there is a phased regularity. In contrast, type-2 fuzzy sets lend themselves to applications in which there is a high degree of uncertainty, specifically situations where there is uncertainty in multiple dimensions [29].

Definitions Complex fuzzy sets in both their forms differ from type-2 fuzzy sets in terms of definition. However there is equivalence between the definition of a pure complex fuzzy set and that of a type-2 fuzzy set.

Structure Structural similarity is apparent between these three-dimensional fuzzy sets. Complex fuzzy sets are represented by a line in 3D, and type-2 fuzzy sets by a surface in 3D. However a surface is a generalisation of a line. The similarity between pure complex fuzzy sets and type-2 fuzzy sets is striking, as both are mappings from the domain onto the unit square.

Operations Standard complex fuzzy sets were not compared with type-2 fuzzy sets as regards inferencing operations. A comparison was made of type-2 fuzzy inferencing operations with those of pure complex fuzzy sets; an isomorphism was shown not to exist.

Overall, the differences outweigh the similarities; despite their shared three-dimensional nature, complex fuzzy sets and type-2 fuzzy sets are two distinct entities, constructed for different purposes, and with different behaviour mathematically.

Complex Definition of a Type-2 Fuzzy Set A definition of type-2 fuzzy sets in terms of complex fuzzy classes has been presented. This can be used to propose alternative union and intersection algorithms for complex fuzzy sets using the corresponding join and meet of the related type-2 embedded sets. This would have the advantage of being supported by Zadeh's Extension Principle [2] that generalises the union and intersection of type-1 fuzzy sets to other types of fuzzy sets such as type-2 fuzzy sets.

Type-2 Complex Fuzzy Sets As further work, combine the (standard) complex fuzzy set with the type-2 fuzzy set in defining the hybrid *type-2 complex fuzzy set*. Investigate the application of this new form of fuzzy set to the analysis of data which is both seasonal and uncertain.

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