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A Scrutiny on Representing – a new view on Meaning and Reasoning

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"And maybe some of this stuff counts as philosophy after all. The form of a philosophical theory, often enough, is: Let's try looking over here"

Jerry A. Fodor [1]

Part One

I Some previous considerations on representing

1. All we intellectually perceive are but representations, and it either if the perception corresponds to a physical reality, or to a purely mental one done in the brain [2]. In the first case (like, for instance, a simultaneously visual, olfactory, and oral perception of a forest), the perception comes from what the senses send to the brain and is processed and represented into it, and in the second (like, for instance, the mental perception of an old and not currently present picture), comes from a former perception stored in the memory, redirected by it into the brain, and even represented in a mentally visualized blurred form because of, perhaps, the errors caused by the own intrinsic limitations of the physical systems the brain is composed of. The particular case of *thinking* (a natural phenomenon, whose study corresponds to Neurobiology), and called *reasoning*, can be done thanks to the brain functioning that, by itself, constructs such representations from the received perceptive inputs. Notice that 'thinking' is larger than 'reasoning', and comprehends more than it, and as it is, for instance, imagining; nevertheless, among those external manifestations of thinking that are observable by everybody, both language and reasoning are paramount. To follow the *Law of Parsimony*, what is here avoided is the (different) subject concerning motor reactions to external or internal perceptions; consequently, situations like those shown by, respectively, experienced or beginner musicians (the first play their instrument without any apparent thinking on it, but the second seems to need such thinking), are not taken into account. Only visual, olfactory, tactile, oral, and those imagined representations that can be called the intellectual ones, will be considered as background. They are, basically, those with which our brain contacts with the world, and from which we try to comprehend it through other representations whose adequacy to avoid miss-representations only can come from its possible, but not unique, checking with the corresponding involved and external reality. This is what Science does, by trying to domesticate both the world and the mental perceptions towards studying the behavior of facts, to understand them and to foresee its evolution; Science tries to intellectually colonize reality, and most of Science is of a dynamical and evolutionary character. After the primary representations automatically done by the brain's systems of neurons, the intellectual study of physical or virtual phenomena do require of new representations that, in many cases, are of a symbolic form created by people thanks to their reasoning capabilities. When the physical world was only perceived at naked eye, people did have a comprehension of it very distant from the current one reached thanks to artificial sensors increasing the potentiality, and enlarging, what can allow the natural ones human beings are

endowed with. In the old times, and since geometrical knowledge was basically bound to spheres and circles, some regular polygons, etc., philosophers did build up mathematical models of the Earth and the visually known universe at naked eye, with such entities for explaining both what is in the Earth's surface, and the apparent movements of the then known sky bodies. This historical example shows, like many others, the relevance of the previously acquired and, let's say, 'theoretic knowledge', for representing something.

Anyway, we have to stress that there is no immediate way of faithfully representing reasoning but with language; there can be thinking without reasoning, but not human reasoning without language.

What is perceived seems to ask, more or less soon, for a rational understanding of it and that only can be reached by means of new representations over those the current sensors, the observations that have been made and the acquired knowledge, allow to obtain. Notice that it was the Galileo's use of telescope what produced, after knowing that the planets movement around the Sun can be faithfully represented by elliptic orbits, a first progress towards Newton's mathematical representations, only reached after a previous multitude of practical observations, and once differential calculus was introduced. The laws of Newton constitute a first mathematical model of gravitation that did hold for several centuries, and show that scientific knowledge is never definitive, but always submitted to refinement and falsification [3]. Some kind of (a softer) parallelism can be established with the passing from hieroglyphic writings to alphabetic ones, and from Roman to Arabic numbers; all that meant an enormous progress not only for commerce but, especially, for knowledge, and like Newton's laws also did. It reflects that representations are never unique, that for each problem several symbolic representations can exist and that choosing the most adequate of them is part of the success for acquiring good and fertile knowledge; that is, with relevant practical applications. This is a goal mathematics help with the aid of the safest available reasoning, the formal deductive one, to which is attributable the 'unreasonable effectiveness of mathematics in the Natural Sciences', to quote a famous paper's title by the Nobel Prize Eugene Wigner [4].

With some level of exaggeration, it could be said that intellectual progress is continuously generated thanks to fiction; the human capability of creating fictions, telling and talking on them thanks to abstraction, conceptualization and language, is what allows knowledge, its refinement and perfecting towards obtaining the best understanding than currently possible on something worrying humans. The history of human thinking could be seen as a kind of fight between what is 'actual', truly contrastable, and what is 'ideal', imagined; science mixes both aspects by starting from both the real

and the ideal, but being always based in the first on which ‘ideas’ should not only relay, but be checked against.

Without guessing it is actually difficult to imagine how people could be able to capture and understand anything; representation, and in particular that of a symbolic character, is among the great helps humans can count with, in the somehow mysterious way of finding the fertile answers that, sometimes unexpectedly, appear in those processes towards comprehension. Trying to representing guessing, or conjecturing from its expression in natural language, seems to be an interesting task, and this paper just considers reasoning from what is described in such language; additional abstractions, like that of taking into account ‘logical’ propositions, are here avoided.

2. Representing is, in general, a powerful instrument (also, and maybe, natural and unavoidable) that helped, helps and will continue helping mankind, and not only for a scientific or philosophical purpose but also for any everyday one, like it is the analysis of a restaurant’s addition; up to now, it seems to have been essential for capturing the most deep nuances of reality, for transforming reasoning into knowledge, and for applying it to praxis. Representing knowledge of the diverse types is a relevant subject, an important task. Even reasoning itself , including the inductive one, can be done with the help of mathematical representations, and like it was in the mid of XIX Century the currently called Boolean algebra [5] for the reasoning with precise concepts, or it is today Zadeh’s Fuzzy Logic [6] for the reasoning with the imprecise ones. To reach faithful representations is an important part of any scientific endeavor. Even a list for remembering pending tasks is a sort of representation, and one should ask her/himself on what does not actually come from a representation.

Basically, a representation of something is nothing else than a relationship established between objects in two universes of which the first is, for instance, either an external physical reality, or a mentally internal one, and the second a collection of brain’s neural systems and with which the image, picture, or representation constituting a first perception of something is (automatically) built up. Representing is, indeed and actually, an important and, perhaps, unavoidable tool for understanding the natural phenomenon of thinking and, in particular, of reasoning. Faithfully representing the reasoning is especially important in view of its mechanizing by computers, and, for instance, as it is in the research field of mathematics where much of the job of creative mathematicians lie, for instance, on characterizing some abstract mathematical structures by means of algebraic or topological) representations with which such structures can be studied by employing the knowledge coming from these parts of mathematics.

The linguistic mechanizing of ordinary, non deductive reasoning, is basic for arriving at computers being able to reason like people do, since we mechanize language with the very use of language, what means that language is not just our study object but at the same time the instrument with which we undergo such study. In this case language is at once an object and a means. This constitutes a very far away goal, called the 'Gordian Knot of Artificial Intelligence', and to undo it new ways of seeing at language, thinking, and reasoning, as well as to represent them, seem to be required. In the authors' view, what is required is a scientific domestication of those subjects, of which only thinking strictly corresponds to the Neurosciences, although some kind of representations are also necessary to, for instance, interpreting images obtained by the current non-intrusive methods like it is TAC. In this view, natural language and ordinary reasoning deserve forms of a new kind of a scientific/experimental research also supported in specific representations (included mathematical modeling) and computation (also requiring mathematical representation). The 'join reality' constituting the ground of this paper tries to refer, is the given by the intermingled natural phenomena of natural language and commonsense reasoning.

Indeed, some mental representations can be translated into symbols and mathematical terms, by means of correspondences, relations, functions, and mappings that, in addition, can be operated with the corresponding calculus developed by mathematicians. Mathematics is itself full of symbolic representations that, in particular and for instance, are used in Physics to represent the phenomena related to matter and energy; for instance, Hilbert Spaces allow to represent those phenomena observable and quantifiable at the Quantum level, as well as to foresee its behavior, and Orthomodular Lattices are but a pass ahead to obtain general enough representations allowing a best understanding of the reasoning with Quantum elements. Riemann spaces are also a representational device important for looking at the universe through the lenses of General Relativity, in some form like graphs are representations allowing the study of concepts.

Actually, the help of mathematics to science is done through sequential processes of representing; from the previous mental representation of either physical, or virtual objects, up to a representation of a sometimes symbolic character to, finally, a synthetic representation in purely mathematical terms. Science and Technology exist thanks to a multitude of sequentially mixed representations; up to some extent, science is an 'art' of the best conjecturing, refuting, reasoning by analogy or metaphor, and representing; like also painting, music, poetry, sculpture, and literature, exist thanks to the possibility of conjecturing, refuting, doing analogies, etc., and finally 'representing' what is the case in either a material, or a symbolic form. At the end, and since also conjecturing and refuting can be analyzed by

representations done thanks to some mathematical models, representing is indeed the kernel of what really matters, and even if what is the final representing step does not exactly coincide with the first step, with what is somehow perceived.

We should not forget that, since natural language, during the process of representing, has to adapt itself to the pressure of its own capacities and our representing goals, it never finishes its work being ‘as natural’ as it had began, because at the beginning it was a language still not committed to specific tasks.

The difference between ‘representation’ and ‘representing’ consists, basically, in that the first is but a name of an abstract concept referring to any representation, but the second just names those (practical) processes trying to reach ‘a representation’ of something previously given. Of course, like in mathematics no generalization is accepted unless at least two particular instances are previously known, and since without at least two previous instances a concept either actually cannot be taken into account, or it comes to just naming a single think, it can be better referring to representing than to representation; in any case, here and from now on, both words will be indistinctly used even if without representing, without actual representations, the abstract and general concept of ‘representation’ cannot exist. In addition to that, we might say with Ludwig Wittgenstein, that ‘a concept is the technique of using a word’. The word naming the concept is but a linguistic label for it, and even if such label does not always correspond with a single concept; anyway it must be recognized that a good concept endows something going further from its instances, and that allows its managing for further reasoning.

Most symbolic representations are like the map of a territory in which not all its places appear, but that allows for a pretty good orientation towards finding any place actually existing. It does not mean, for instance, that mathematical representations, or models, can explain or foresee all phenomena; inside such models there can be questions proven as not deductively decidable, like it can be places in the territory that cannot be found in a map. Like a new and more complete and precise map can help to find such places, a new model also can help to explain or to foresee such phenomena; in fact, and along history, it is not rare the sequencing of models, or representations, each one augmenting the level of explanation or of foreseeing of the former. For a good enough mathematical representation it is basic the adequate selection of a ‘natural frame’ of representation for those phenomena in course of study; this is the case, for instance, of the Euclidean space and the classical differential calculus Isaac Newton took, and that Albert Einstein did need to change to Riemann spaces and tensor calculus for

modeling his General Relativity that enlarged the Newton theory of Gravitation, as well as offered a new view on the macro-world's structure. In any case, what should be remarked is that what cannot be expected is a complete explanation coming from a current model of representation, but that and as history shows, what just can be expected is to attain a more plausible level of explanation, coming from sequences of better and better models made in frames of representation each time seeming more adequate to the problems in question. It is not to forgot that models do not constitute the full reality, but just representations of it; indeed our representations are just ways of describing possible explanations. Representation should be seen inscribed in a dynamic and non static process of representing; up to some extent like an Architect's mockup, into which refinements can be added continuously up to a final moment in which it offers to the designer all it is able to show. There are in history a lot of examples in which knowledge did advance through a sequence of representing models each time more general and comprising, as instances, the former ones. It should be remarked that some models allow to reach some practical successes even if those models are latter on proven incorrect. This was, for instance, the case of the geocentric model of the solar system that, even proven incorrect, and with the complications coming from the epicycles, allowed to determine pretty well the positions, at naked eye, of the planets. In any case, as it has been said, models cannot be finally accepted without its confrontation with the corresponding reality; for instance, Newton's model of gravitation and the discrepancies in the planetary system, was not well established until Laplace did find the mathematical way of checking it with mountains of real data, and Einstein's model concerning the mass of light until Eddington did his famous observation thanks to a solar eclipse. Actually, a scientific model is but a synthetic representation of some phenomenon, somehow recognized as a real one, and done with symbols translating the basic elements and relationships among them that appear in it. For arriving at a model, 'mental experiments' (like those Einstein imagined with a lift before writing his paper on the General Relativity) are perfectly acceptable as a part of the previous, and necessary, reflection on which is the problem, and how it can be solved; but to finally accept the model as an adequate one, it should be checked against the corresponding reality. On the contrary, the model just means a case of the metaphor 'constructing castles in the air', that is, it is simply an imaginary model of who ideates it; hence, and in the best case, just valid as a suggestion.

3. In this paper we will try to do a glance at the important and mixed case of symbolically representing natural language and ordinary reasoning in which it is very difficult, provided it were actually possible, understanding the one without the other; both seem to constitute a kind of continuum [7] supporting rationality.

Of language there is the clear fact that we can observe it through the several ways it can be externalized, the spoken, gestural, and written forms. Of reasoning there is also the observable fact that it is basically and usually transmitted and recognized by means of natural language that, basically, deals with perception and reasoning; it is the physical medium on which reasoning can be expressed and communicated either if it is in a spoken form, or is done by gestures, or is written. All that will be tried to be done by seeing everything from outside; as far as possible from internal, psychological, considerations.

Which is the role of representation in both cases? Before communicating either an elemental affirmative statement, or a large phrase translating a complex reasoning, it is necessary to mentally represent, in some form, what it is intended to be communicated, and to suppose that the receiving captures what the emitting expresses by what is, at its turn, a representation of the statement's or phrases' contents. Finally, these words are to be represented by spoken, written, or gestural signs send out by who is trying to communicate something. It should be distinguished between the brain's neural (physical) structures allowing to built up what is going to be expressed, and their representations by either spoken, written, or gestural expressions; there is always a sequence of several representations of different character. The distinction between, first, the corresponding mental images, and, second, the phrases expressed by spoken, gestural or written signs, should be clearly taken into account. Expressing any reasoning requires a specific enchaining of representations of several characters, even and perhaps of a symbolic one, where the concept of meaning and, consequently, of Semantics, is hidden.

It should be noticed that this paper it not addressed towards a Natural Science view of human life and cognition; for instance, the brain as the 'main central controller' from which the internal correct functioning of the body is commanded, is out of its goal. As it is out of it how the brain functioning makes possible representing mentally something; even the physiological ways under which the brain originates understanding is out of it. All those topics do not correspond to what the authors would like to stress by reflecting on, and this paper will just try to offer some hints for a first scientific and philosophical reflection, in the case of language and reasoning, on the relevance of what is named as 'the representation', 'a representation', and 'representing'. It will only consist in some first thoughts on representing; a topic into which very few thinkers did actually devote their attention.

Remarks

a) Only orally, written or gestural expressed ‘statements’ are here taken into account, but not the logical and philosophical idea of ‘proposition’ behind them [8]. The reason for avoiding the concept of proposition is that it can be seen as an abstraction, supposedly containing all its instances that can be translated by particular statements, referring to a common meaning of all of them, and related to some kind of a universal logical form of it. In what follows, and as it is typical of Natural Language and Commonsense Reasoning, everything is considered to be context-dependent and purpose-driven; a view under which meaning is not a universal characteristic of particular linguistic statements. Following Ernst Mach [9], what can be said on concepts like that of ‘proposition’, is that, if not empty, they charge sometimes with a history both delimitating and allowing its use.

b) Some uses of the word ‘representation’ are not considered, like it is, for instance, the case of ‘theatre representation’, or the seeing of literature as an ‘intelligent representation’ of life, etc., and even if all this is not too far from the representation’s concept sustained here.◊

4. One of the two subjects on which this paper tries to shed some light is the representation of the modalities of commonsense reasoning, in which both the ideas of the deductive, and the non- deductive, reasoning are present. In this line, it is perhaps interesting the distinction that will be made here between deduction and induction. Induction, sometimes called the ‘scandal of philosophy’, is the kind of non-deductive reasoning encompassing those inferential processes directed towards expanding knowledge. In fact, the enlargement of the initial information supplied by some premises cannot be done by formal deductive reasoning [10], but requires speculating on them. In particular, the creative speculations, introduced by the first time as the conclusions of the type of guessing or conjecturing that, being non-monotonic and not deductively related with neither the conjunction of the premises, not with its negation, allow for ideas going further from the premises [11]; new ideas, in sum.

Part Two

II On a Representational basis for (a scientific) semantics

This section is devoted to introduce a domestication of meaning, that is, a symbolic frame representing both its primary spring, and how to reach a secondary spring completing 'meaning'. Meaning appears as something of a twofold character with both a qualitative and a quantitative component.

1. When a receiver captures what an emitting sends to him/her, and, in some form adequately reacts to such message, it can be supposed that s/he did have understood it: The receiver should have captured the meaning of the message, or some significant part of it, and the correctness of such understanding can be checked by the taken, or decided, action the receiver does; it can be said that messages transmit information, that statements or phrases contain something with which they are understood. This something is meaning, and Semantics deals with meaning. Understanding goes further than the simple (grammatical) description of the statement; the semantics of a statement just refers to its meaning, and it concerns more than the grammar on which the statement is correctly constructed, concerns the environment surrounding it, the context on which it means something. In different contexts grammatically equivalent statements can have different meanings. For capturing the meaning of a statement, considering its grammatical form (its syntax) is necessary, even if incorrect forms are also able to transmit some meaning. The universe of discourse and the context at which the statement refers to, how the parts of the statement are actually used in it, and also its purpose for such use, are the basic items for comprehending its meaning [12]. All those components of meaning should be taken into account for capturing it, and for trying to mathematically representing it.

2. Under which guidelines meaning can be mathematically represented? That is, how can meaning be tried to be mathematically domesticated in some profitable way? The following example considering a basic single word, a predicate, can be in order. Let it P be the name of a property shown by the elements of a universe X (a set), that what P names can be recognized in the elements of the set X; P is a predicate acting on X. Such recognition can be done once it is perceived if any x in X verifies, or not verifies, the property named P, and even if both are just done up to some extent. That is, by empirically recognizing if, for all x in X, the linguistic statement 'x is P' can be safely asserted, and up to which extent it can be asserted; it implies to accept that meaning is not something static and universally associated to each P. Indeed, the meaning of P in X is carried by all the elemental statements 'x is P' that could be taken into account.

Once all these elemental statements, at least potentially, are asserted, and to well comprehend what P asserts in X, it should be empirically recognized the relationship established among the elements of X, by [13]:

x is *less* P *than* y,

symbolically represented by

$$x <_P y \Leftrightarrow (x, y) \in <_P,$$

and, perhaps, adding the non surprising supposition that the binary relation $<_P \subseteq X \times X$ (less P than), at least enjoys the non rare reflexive property,

'x is less P than x' $\Leftrightarrow x <_P x$, or $(x, x) \in <_P$, for all x in X,

by understanding that 'less P' contains 'equally P', and with which it is sure that the relation $<_P$ is not empty. Such symbolic relation $<_P$ shows how P primarily varies qualitatively in X; without recognizing it, the knowledge of the behavior, or use, of P in X is not well captured. Provided only a part of $<_P$ is known, such use is just partially captured.

Provided that, instead of the relationship 'less P than' what is only empirically recognizable is the smaller relationship '*equally* P *than*', that is

x is equally P than y,

symbolized by: $x =_P y$, that, by defining

$$x <_P^{-1} y \Leftrightarrow y <_P x \Leftrightarrow x \text{ is } \textit{more} \text{ P } \textit{than} \text{ y},$$

implies

$$=_P = <_P \cap <_P^{-1},$$

and shows that $x =_P y$, means that x is less P than, and also more P than y; the only that is recognizable is that relation $<_P$ coincides with $<_P^{-1}$, that both $<_P$ and $<_P^{-1}$ collapse in $=_P$.

In the first case ($<_P \neq =_P$), P is said to be a (linguistically) *regular*, or imprecise, predicate in X, and in the second, $<_P = =_P$, that P is (linguistically) *irregular*, or precise, in X. In the first, it is perceptively recognizable that x is P up to some extent, and in the second just that x is, or is not, P, but nothing else. For instance, 'tall' is a regular predicate in a big population since statements like 'John is less tall than Paul' can be recognized, but 'odd' is irregular among natural numbers, since all of them are equally so, or are not; 25 and 1053 are equally odd, and 24 and 756 are equally not odd. At each case, the respective graphs $(X, <_P)$ and $(X, =_P)$ do represent the qualitative meaning,

or *primary use*, of P in X. The names 'irregular' and 'regular' refer to the words in natural language, but not to those in any artificial one.

Notice that, if existing, the maximal elements in the graph $(X, <_P)$ are those x in X for which there is not any $y \neq x$ in X, such that $x <_P y$, and the minimal elements are those z for which no $y \neq z$ such that $y <_P z$ exists in X. Notice also that 'less than' and 'more than' are opposite concepts, and, hence, a maximal x is an element for which there are not in X other elements being more P than x; analogously, for a minimal z there are not other elements being less P than z. Provided such elements exist, some arcs of the graph end in maximal elements, or start from minimal elements. Both relations $<_P$ and $<_P^{-1}$ complete, at a first stage of perception, what can be said on the primary linguistic use of P in X, and in the case of a precise predicate they just collapse in the relation $=_P$. Is in this sense that precision, or linguistic irregularity, appears as a limiting case of imprecision, or linguistic regularity; an assertion that corresponds with what can be observed in natural language. For instance, it suffices to look at any dictionary for observing the comparatively very few precise words that, naming a predicate, exist in it, that is, words whose use is *defined* by an 'if and only if' description, or proper definition, like they are those words concerning almost all mathematical concepts. Notice that this does not presuppose that the definition is unique; for instance, and in general, mathematical predicates can be defined in several, and obviously equivalent, forms. The use of the regular predicates is just *described* by some statement, or some rules, for using them and that, depending on the context in which they can be used, cannot characterize a single use of the predicate.

3. Once the graphs $(X, <_P)$, or $(X, =_P)$ are established, it yet lacks to know up to which extent each x in X shows the property named P, a problem of 'size' that is of a new and quantitative character and, hence, lies on the effective possibility of measuring such extent, let's say the amount of P shown by x. Before such possibility, the only that can be said on P, provided is known that such graph exist, is that P is *measurable* in X, and yet preserving the term *effectively measurable* for a next step when a measure of such amount is known.

All that is similar to what happens with so familiar concepts like 'length', 'surface', 'volume', 'randomness', 'electrical current power', etc., and this similarity comes from knowing when, for instance, a surface is smaller or bigger than another one. It is from the knowledge of the corresponding empirically perceived graph $(X, <_P)$ that is possible to define the concept of a measure of the extent of P in X. Of course, measures can enjoy different properties among different kind of objects; for instance, if 'length' can be seen as being clearly additive, it seems at least dubious that, in general, 'vagueness' could be so.

To present a general definition of what a (numerical) measure is, let [a, b] be a closed interval of the real line endowed with the usual linear order ≤.

Definition. A mapping m: X [a, b], is a *general measure* of the extent of P in X [14], provided:

1. If $x <_P y$, then m (x) ≤ m (y),

2. If z is a maximal in the graph, then m (z) = b,

3. If z is a minimal in the graph, then m (z) = a.

The interval [a, b] can be, in principle, the unit interval [0, 1], or the interval [-1, +1], or the interval [0, +∞] of the positive real numbers, etc. Nevertheless, and since it is an obvious fact that elements in X not comparable under $<_P$ can exist, linearly ordered intervals show the trouble that for any two elements x and y, such that they neither verify $x <_P y$, nor $y <_P x$, one of its values under a given measure should be greater than the other; that a general measure does not preserve incomparability under $<_P$. From now on, 'general measure' will be shortened to 'measure', even if remarking that the concept of a general measure is the common core all measures should verify to be named so, and although they were defined by means of properties, or axioms, hiding but actually verifying, the former three. It is the case, for instance, of probabilities, in whose definition is hidden the first property.

Notice that what any measure actually preserves is the relation 'equal than'. Indeed, provided it is $x =_P y$, or both $x <_P y$ and $y <_P x$, property 1 implies m (x) ≤ m(y), and m(y) ≤ m(x), that is, m(x) = m(y). Equally P elements enjoy an identical measure's value.

It should be noticed that property 1 is essential. For instance, provided it is perceptively and clearly recognized that 'John is less short than Paul', we will consider absurd if the measure of the second is 185 cm, but that of the first is 170 cm: The result 185 = m (height (Paul)) > m (height (John)) = 170, immediately forces a total loss on the confidence in the instrument (centimeter) being used to get the height of people; it will be against evidence! Concerning properties 2 and 3, notice that they are only applicable provided either maximal elements, or, respectively, minimal elements, do exist; there are indeed graphs without maximal or minimal elements that, if existing, can be seen respectively as prototypes and anti-prototypes of P in X. There are also graphs with a single maximal (the maximum), or graphs with a single minimal (the minimum).

All that is but towards showing that if general measures could be defined, in principle, for all measurable predicates, it neither does imply that a measure could be effectively specified or constructed, not even that the

measure could be a single one. The specification of a measure could require, at each case, either acquiring more information on the graph, or adding some plausible hypotheses on the behavior of the given P in X. At this respect, the elemental example of the predicate S = small in X = [0, 10] is illustrative.

Since in [0, 10] it can be easily agreed that 'x is less small than y' (or that 'y is more small than x') if and only if it is $x \leq^{-1} y$, or $x \geqq y$, it follows $<_S = \leq^{-1}$, that is, the qualitative use of S is given by the graph ([0, 10], $\leq^{-1}$); S is measurable in [0, 10], and there are only the maximal 0, and the minimal 10 in the graph. Hence, its measures are those mappings m: [0, 10] [0, 1] such that:

1. If $x \leq^{-1} y$ ($y \leq x$): $m(x) \leq m(y)$,

2. $m(0) = 1$,

3. $m(10) = 0$.

That is, the measures of 'small' in [0, 10] are (all) the decreasing functions [0, 10] [0, 1] joining the points (0, 1) and (10, 0) in R^2; thus, for specifying one of them, it is necessary to count with more information on how S behaves in X. For instance, provided we can plausibly add that the measure should be linear, $m_1(x) = ax + b$, then it depends on the two parameters a, b of which and to be decreasing, it should be $a < 0$. Then, from 2 follows $b = 1$, and from 1 follows $a = -1/10$ giving the decreasing mapping $m_1(x) = - x/10 + 1$; hence, this is the only possible linear measure that can be specified for S. If, also for instance, what can be plausibly added is that the measure should be quadratic, $m(x) = ax^2 + bx + c$, it just follow $c = 1$, $100a+10b+1=0$, and the condition $dm(x)/dx = 2ax + b \leq 0$ (*) for all x in [0, 10] (of having a negative derivative provided it is, like in this case, a derivable function), assuring that m is decreasing in [0, 10], with which there are yet the two lacking parameters a and b, and additional conditions should be added to determine them: There is not a single quadratic measure, and all of them are defined by

$$m(x) = ax^2 + bx + 1,$$

with the parameters a and b verifying:

$$100a + 10b + 1=0, \text{ and } 2ax + b \leq 0, \text{ for all } x \text{ in } [0, 10].$$

Among a typical additional information, there is, for instance, $m(5) = 0.7$, that facilitates the equation $25a + 5b + 1 = 0.7$, and then $b = - 0.02$, $a = -9/250$, values that verify inequality (*) for all x in [0, 10]: Hence, the corresponding quadratic specified measure is $m_2(x) = -9x^2/250 - 0.02x + 1$, and, also for instance, the square of the linear one, $m_3(x) = (-x/10 + 1)^2 = x^2/100 - x/5 + 1$, is another quadratic measure that could be specified, and for which it is $m_3(5) = 0.25$. Many possible quadratic measures of S in [0, 10] exist.

It should be noticed that there can be elements with measure zero or one that are neither qualitative prototypes, nor qualitative anti-prototypes, respectively; for instance a different measure of 'small' in [0, 10] is the piecewise function m_4 given by:

$m_4(x) = 1$, if $0 \leq x \leq 1$;

$m_4(x) = 0$, if $8 \leq x \leq 10$; and,

$m_4(x) = (8-x)/7$, if $1 \leq x \leq 8$,

with which a lot of elements with measures 1 and 0, and different from the prototype x=0, and the ant-prototype x = 10 appear. These elements, not existing with the three former specified measures, can be called working-prototypes and working-anti-prototypes, respectively, and are coming from the new, additional and contextual information taken into account for specifying the measure m_4.

This example with 'small' makes clear that the qualitative use of a predicate is not, in principle, sufficient for specifying a single measure of its qualitative meaning, and that additional information is necessary. The full meaning of a word always depends on the universe in which it is used, on the context of its use in spoken, oral, or gestural form, and on the purpose for it. Even if the representation of the universe is without strong troubles, those of the context and the purpose are (usually) laborious.

What happens with irregular predicates? Since given any two elements x and y, with P irregular it only can be either $x =_P y$, or not, the set X is partitioned into the set containing all the elements that are equally P than another one, and its complement, that is, the set of those that are isolated except from themselves; the first are maximal, and the second are minimal elements. Obviously, the first do have measure m constantly equal to one and, for the second its measure should be equal to zero: Irregular predicates not only are measurable, but have a single measure assigning the value 0 to the 'isolated' elements, and the value 1 to the rest. All irregular predicate specifies in X the single subset $m^{-1}(1)$, consisting in those elements with measure one. Contrarily to the regular case, no more information than its definition is necessary to specify the set $m^{-1}(1)$ defined by an irregular predicate through a very strong 'if and only if' condition. For instance, P = odd in $X = Z^+$ = positive integers, specifies the set of numbers with measure 1, $m^{-1}(1) = \{2n+1;\ n \in Z^+\}$. It should be noticed that such functions do have jumping points like, for instance and in this example, there are points like x = 1, and x = 5, where the function jumps from 1 to 0 before x = 2 and x = 6.

4. Each triplet $(X, <_P, m)$ can be seen as a representation of the *full meaning* of P in X, and it is clear that a regular P has not a single meaning in X

but that all of them depend on the context on which its linguistic use is inscribed in, as well as in the purpose driving such use. The full meaning of a predicate is a matter of design.

A final and illustrating example can be done by the regular predicate P = probable, used in a Boolean algebra Ω of subsets A, B, C, …, of a universe X, in which it is usually represented 'A is less probable tan B' as 'A is contained in B' ($A \subseteq B$), with the back-thought that as less elements the subset has, less probable it is. Then the qualitative meaning of P in Ω is the graph $(\Omega, \subseteq)$, with the single maximal X, the single minimal Ø, and with which a measure of 'probable' m: Ω [0, 1], that is, of the statements 'A is probable', 'B is probable', etc., should verify:

$$1)\ A \subseteq B \Rightarrow m(A) \leq m(B);\ 2)\ m(X)=1, \text{ and } 3)\ m(\emptyset)=0.$$

Provided it could be supposed that m is a Kolmogorov's probability, from its laws $m(A^c) = 1 - m(A)$, and $m(X) = 1$, follows $m(\emptyset) = 1 - m(X)=0$, and from its additive law $m(A \cup B) = m(A) + m(B)$, if $A \cap B = \emptyset$, follows that since $A \cup B = A \cup (B \cap A^c)$ and, since $A \subseteq B$ is equivalent to $B = A \cup (B \cap A^c)$, where it is $A \cap (B \cap A^c) = \emptyset$, it is $m(B) = m(A) + m(B \cap A^c) \geqq m(A)$: m verifies the first property of a general measure. Hence, and actually, Kolmogorov's probabilities are general measures, and, as it is well known, at each case the most suitable for the current problem should be specified since its laws don't specify a single probability (for instance, and just in the simple case of throwing a coin, it is not possible to specify the probability of getting 'face' without information on both the current coin's constitution and the landing surface). A coin with both probabilities equal to ½ is just the 'ideal coin'. Nevertheless, there is no clear enough evidence showing that all measures of the predicate 'probable' should always be specified as a Kolmogorov's probability, and specially, if 'probable' is not applied to crisp events, but to imprecise concepts with which it is difficult (if possible) obtaining frequencies. This is, for instance, what happens between the measure of 'obtaining six in throwing a dice is probable', and that of 'Candidate X will win the election is probable'. To some extent, these questions constitute a part in the controversies between the Bayesians and the frequentists [15], and also of the Bayesians with those researchers following Zadeh's ideas on fuzzy probability [16]. Fuzzy probability concerns measures of probable whose values are, instead of real or complex numbers, membership functions and deals with linguistic examples like m ('John is rich is probable') = high, representing by membership functions the two fuzzy sets respectively labeled 'rich' in some population, and 'high' in the interval [0, 1]. Obviously, numerical probability can be seen as a particular case of the Zadeh's one by just identifying numbers r with the functions $f_r(r) = 1$, $f_r(x) = 0$ if $x \neq r$, the characteristic function of the singleton {r}.

Remarks

a) It should be pointed out that each measure of P in X is a membership function of the Zadeh's fuzzy set labeled P [13]. Hence, each 'designed' good membership function can be seen either as a measure of the qualitative meaning in X of the linguistic label, or as approaching one of them. At the end and as Zadeh always pretended [17], they are nothing else than a representation of the full meaning of P in X that, understood by him as 'extensional meaning', requires to be measured, and that in praxis should be designed with the best approximation than possible. This is the reason for which usually a membership function is but an approximation of a measure; this is so because the designer does not usually know all the qualitative meaning of the linguistic label.

Of course, classical sets appear, from several points of view, as 'degenerate' fuzzy sets. In a first place, since the relation $<_P$ collapses, or better, degenerates, in the relation $=_P$, and in a second place since its membership function called, in set theory, its characteristic function, is unique and only takes the values in the short subset {0, 1} of the unit interval. Its values characterize the behavior of P in X, that is fully specified by the single subset $m^{-1}(1) \subseteq X$. Consequently, the specification of an irregular predicate is a unique subset of X whose membership to it is defined by 'if and only if conditions'.

On the contrary, the specification, or representation, of a regular predicate is not unique since its linguistic description is almost never complete, is not facilitated by an 'if and only if' definition. To specify a fuzzy set with a regular linguistic label P it is necessary to design a measure of P in X, and it is important to realize that fuzzy sets are not only, as Zadeh likes to say [18], a matter of degree, but also a matter of design [19, 20]. A design that will be better as more approximate to a measure can result the membership function the designer obtains with the best available information at his/her hands on the behavior of P in X, and that not always can be all the graph $(X, <_P)$. Fuzzy sets are situational entities [12], that is, their linguistic label is context-dependent and purpose-driven.

b) There remain some topics not yet well described, and consequently not still well understood. One of them is the distinction between fuzzy set and membership function, two concepts so often confused but that cannot coincide since the membership function approximating a measure is not unique, and the fuzzy set is usually understood as a single, but blurred or cloudy entity associated to P, and enjoying some extensionality, or 'size'. If accepting, following Quine [21], that 'there is no entity without identity', it seems adequate any intent for identify what a fuzzy set actually is. Let us advance a possible explanation eventually able to take into account such

question. It is easy to recognize in language that in some form all predicates P collectivize in the universe of discourse, that is, generate the 'Ps *linguistic collective*' [14]. For instance, 'small' generates In X = [0, 10] the linguistic collective of 'the small numbers between 0 and 10'; 'young' generates in X = London's inhabitants the collective of 'the young Londoners', etc. These *collectives*, well anchored in language, are nothing else than what is usually called a fuzzy set. Linguistic collectives, or fuzzy sets, are a kind of cloudy entities of whose mathematical representation is not known how it can be obtained from the pair (X, P), since it depends on both the context and the purpose for the use of the predicate generating the collective. Usually, there is not a general mathematical description of them even if, in some cases, like it is that of 'small' in [0, 10], it is possible to identify the qualitative meaning and describe their measures by a finite set of properties. It seems as if these cloudy entities do project a shadow each time at least a context illuminates them, and allows to reach a measure; such measure plays the role of a shadow. In any case, what it does not exist is a general representation of, for instance, 'small' independently of the universe of discourse, the context, and the purpose. Linguistic collectives don't seem to admit a universal representation, nor even in its same universe of discourse, and only those that are generated by an irregular predicate admit a universal representation, that they are in a single state, although restricted to the universe of discourse in which they are defined by means of a 'if and only if' definition. It seems that collectives, fuzzy sets, can only be specified, represented, by each one of its measures, that is, by the triplets $(X, <_P, m)$ that, then, can be seen as a 'state' of the collective. If a collective can exist in many universes of discourse, only can be represented at each one of these universes once the context of the use and the purpose for it can be taken into account; when a state of it is reached. Indeed, if it is still an open question the direct representation of linguistic collectives, people easily manages the linguistic collectives, or fuzzy sets; linguistic collectives mean something people captures.

c) In language, predicates often migrate from a universe to another; for instance, 'small' migrates as 'short' from a numerical universe to a human population, as well as it does 'big' to reflect 'tall', or 'around number sixteen' with respect to 'close to sixteen people', etc. In cases like those and for instance, a measure for 'short' is easily obtained thanks to the height of each person and a measure for 'small'; for instance and provided it is considered $m_{small}(x) = -x/10+1$, if John height is 1.80m, it results $m_{short}(\text{Height(John)}) = m_{small}(1.80) = -1.80/10 + 1 = 0.82$, that allows to say that 'John is tall' is up to the degree 0.82. This transformation can be done, at least, each time the second predicate admits a numerical characteristic related with the first. This kind of a scientific style study of predicates' migration can, eventually, allow at looking the subject of Wittgenstein's 'family resemblances' [31] from a new point of view that, perhaps, can conduct towards its mathematical

domestication [22]. It could mean an interesting way for the analytic study of analogical reasoning through mathematical representations. Actually, it is still an open question.

d) A predicate is *meaningful* whenever a graph $(X, <_P)$ for it is captured or, at least, its existence can be arguably supposed [14]. Provided either it is not known to which specific universe of discourse X a predicate P is applied to, or its qualitative meaning is unknown, it is said that P is *currently metaphysical*. Provided it can be shown either that no relation $<_P$ can exist, or that it exists but is empty, the predicate P is said to be *meaningless* in X. Of course, a currently metaphysical predicate can, later on, become either meaningful, or meaningless, and it should be noticed that the graph or qualitative meaning of P in X, comes to represent what is intuitively perceived when trying to develop a discourse on something: That the meanings attributed to what is being to be said can introduce some order, or hierarchy, among the elements in the universe allowing to clearly exposing the argumentation in course. It can be illustrated by the typical congratulation 'Yours was a well ordered presentation!'. A goal of intelligent discoursing is to introduce some organization among words, some ordering with the goal of clarifying how 'words play with things' in a given situation.

Concerning the names of 'qualitative meaning' for what represents the pair $(X, <_P)$, and 'quantitative meaning' for what represents the triplet $(X, <_P, m_P)$, it should be remarked that there is the possibility of renaming them 'meaning' (Bedeutung), and 'sense' (Sinn), respectively. This, provided it is understood that 'meaning' refers to something only relative to the words, but 'sense' requires considering where and for what words are used.

e) As it has been said, in praxis membership functions are build up with the information on the current use of P in X available to its designer who, usually, does not know all the graph $(X, <_P)$, and, hence, cannot prove that the membership function (f_P) s/he obtained, is actually a measure of the qualitative meaning of P. Most applications are bounded to work, after leaving aside the unknown part of $<_P$, with just the designed membership function f_P, and by supposing it is a good enough approximation to a measure of the meaning; that it exists a measure m such that $|m(x) - f_P(x)|$ is very small for all x in X. In this typical case, the only available relation among the elements in the universe of discourse, is the one given by [13],

$$x <_P^* y \Leftrightarrow f_P(x) \leq f_P(y),$$

with f_P: X [0, 1], the membership function supposedly approaching in X a measure of the meaning of P, and then the triplet $(X, <_P^*, f_P)$ acts as the 'working meaning' of P in X instead of a full meaning $(X, <_P, m)$. Designing a membership function modifies the meaning; in general, it can be said that

more (and, usually, quantitative) information modifies meaning. Measures, membership functions, come from the available information on the qualitative behavior of P in X, but they modify the previous and qualitative information.

Provided f_P is a pretty good enough approximation to a measure in the sense of, at least, preserving its first property ($x <_P y \Rightarrow f_P(x) \leq f_P(y)$), it is obvious that $<_P \subseteq <_P^*$: The working meaning contains the qualitative meaning, enlarges it. The act of design can add arcs to the qualitative meaning; perhaps not surprisingly, makes it larger. This is a point still deserving to be clarified, but on which it should be remarked that if the just reflexive relation $<_P$ is rarely a total, or linear, one (that is, there are, under it and usually, incomparable, or isolated, pairs of elements), the new relation $<_P^*$ is not only a linear one (under which all pair of elements are comparable), but a partial order. It is so, even if f_P is actually a measure of P in X. The act of measuring in a real line's interval, makes the meaning linear.◊

5. The fact that isolated elements under $<_P$ (that is, no one of them is less P than the other), do have always comparable measures is, actually, some weakness in the measure's definition that, in Science, is usually treated by changing the real by the complex numbers; in Physics, for instance, there is not so rare to count with complex valued measures, as it is the case in the study of electrical current. Hence, it is still possible changing the linearly ordered unit real interval ([0, 1], ≤) where m takes its values, to the complex one **I** = {a + ib; a, b in [0, 1]}, endowed with the partial and non linear order [13]:

$$a + bi \leq a^* + b^*i \Leftrightarrow a \leq a^* \ \& \ b \leq b^*,$$

with minimum 0+i0, and maximum 1+i, in which there are non comparable numbers, like they are 0.5 + 0.7i and 0.8 + 0.5i. Then, measures m could be redefined as mappings, m: X **I**, endowed with analogous properties to the three former for the real valued ones that, then, will result the particular case of m ranging in the subset of those complex numbers in **I** without imaginary part (b=0). This generalization cannot assure that two non-comparable elements under $<_P$ in X will have, always, non-comparable measures in **I** but, of course, opens a window to such possibility.

For what concerns irregular predicates, in which $<_P$ coincides with $=_P$, it is clearly sufficient the real unit interval, or whatsoever totally ordered set with only two different elements, since all elements in $m^{-1}(1)$ are comparable prototypes, and those in $m^{-1}(0)$ are out of consideration since they are anti-prototypes, those that are absolutely not P.

6. To end this section a comment is still in order. For going towards a scientific study of language it is needed to count with numerical measures

that could be firstly obtained from simple, but not blind, observation, and secondly tested by controlled processes of experimentation. This is the way science proceeds, and the explanation of meaning by its (observable) qualitative features, and its quantitative strength, is just a tentative in such direction that should be further continued. For such a continuation there are yet some important but unanswered questions like it is, for instance, the analysis of the qualitative, and the full, meaning of large statements, something that logic does by decomposing these statements in elemental parts joined by connectives when often in ordinary language the meaning of components is only grasped after capturing the meaning of the full statement. Even if large statements are analyzed from its parts joined by connectives, remains the questions of interest, e.g. to know if the design of measures for P and Q can influence those of 'Q and P', once they appear in composed statement like 'x is P and y is Q'. The action of connectives is not so simple and less again universally constant, as is classically considered; connectives are also words endowed with a context-dependent and purpose-driven meaning, whose study should be done through a careful analysis of how they are actually used in natural language. For instance, and since in language 'and' is not always commutative since usually time intervenes, it is an almost open problem to describe $<_{P \text{ and } Q}$, once $<_P$ and $<_Q$ are given in, respectively, universes X and Y, as well as to obtain suitable measures for 'P and Q' in X x Y from respective measures of P in X, and of Q in Y [23].

III On representing ordinary reasoning

1. Basically, ordinary, everyday, or commonsense, reasoning consists in the natural processes generated in the brain that start from some previous knowledge, evidence, or information, and that once represented by linguistic statements called premises, allow to draw conclusions. It does not matter if such statements are expressed to oneself or to another person, and if they either are uttered, or written, or gestured. What really matters is that they are conveyed by natural language; it is the problem of representing natural language and commonsense reasoning that is in the core of the so-called 'Gordian Knot' of Artificial Intelligence. The subject is not only interesting by itself, but also for its possible applications to A.I; if finally successful, a new form of looking at the study of natural language and ordinary reasoning can be, at least potentially, a fertile new field of research.

A central subject in the analysis of ordinary reasoning refers to which types it consists in, and finding suitable formal frames of representation permitting the construction of general enough mathematical models, able to deal with the totality of ordinary reasoning; to scrutiny it, and to study its several types. What follows is nothing else than a first trial to begin with a very general and simple symbolism, helping to content with what is already

done for the study of reasoning without presupposing, but allowing to add , when necessary, some formal laws up to the upper limit of Boolean algebras enjoying the greater than possible number of independent laws. Among the laws a lattice enjoys, and Boolean algebras (lattices endowed with a strong negation) enjoy in particular, there are many that cannot be always supposed in ordinary reasoning; for instance, the identification of a conditional statement 'if p, then q' with 'not p or q', that, suitable in formal deduction with precise statements typical of Boolean algebras, cannot be used in many cases of natural language as it was proven many years ago [24]. Even in lattices weaker than Boolean algebras, like they are the De Morgan ones, the identification p q = p' + q (that is, identifying 'If p, then q' with 'not p or q), does not verify the Modus Ponens inequality (that opens the use of such conditional to deduction): Provided it were $p \cdot (p' + q) \leq q$, for all p and q, it suffices to take q = 0 (the lattice minimum) for obtaining $p \cdot p' = 0$, for all p. Then, it follows that In De Morgan algebras, Modus Ponens forces the antecedent of the conditional to be one of its Boolean elements; the representation of a conditional in the form 'nor/or' cannot be used for all pairs p, q. Analogously, in the setting of all Ortholattices, the Modus Ponens verification by p q = p' + q forces the lattice to be a Boolean algebra [25]. Practically, the interpretation not/or of conditionals is just a Boolean interpretation [25].

The following (new) model is a general enough one, but like all models, its validity should be tested against true ordinary reasoning; meanwhile, it is a kind of mathematical playing. Anyway, and even waiting for such test, the model shows some results that are coincidental with what is known, under the usual hypotheses, in classical crisp reasoning, quantum reasoning, probabilistic reasoning, reasoning under imprecision, as well as it allows to formalize some Popper's ideas in the typical reasoning of Science, on conjectures and refutations [30]. The model allows to prove such knowledge under very weak hypotheses not charging language and reasoning with 'dubious' laws, only necessary for conducting proofs in algebraic structures stronger than the very simple one that is here presented [11. 26].

The authors strongly believe in the methodological power of the old XIV Century principle called the Occam's Razor (or *Lex Parsimoniae*), of not adding more entities than those that are strictly necessary, although complemented with the XX Century Menger's addenda of not adding less than those necessary for reaching significant results [27[. At the end, any representation in a formal frame requires to construct a theoretic framework where both representing the questions, and doing some deductive proofs, can facilitate a search for answers whose validity obviously will depend on those conditions accepted for the goal of proving them. Meanwhile, and under the *Lex Parsimoniae*, it seems better to work under a minimal number of clearly

posed hypotheses, as well as to escape from any psychological aspect and only taking into account what can be observable by all people. A first approach with fuzzy sets and very few laws, was recently done [11,23,26] in the setting of Basic Fuzzy Algebras.

2. Given a finite set of premises $P = \{p_1, \ldots, p_n\}$, previously placed in a relevant ordered sequence, let p the conjunctive statement,

$$p = p_1 \text{ and } (p_2 \text{ and } (p_3 \text{ and } (\ldots, \text{and } p_n)\ldots),$$

called the résumé of the premises. Notice that were the linguistic conjunction 'and' both commutative and associative, then the parentheses can be avoided, and the ordering of the premises becomes irrelevant. In what follows, the symbol (·) will be kept for representing 'and'; for instance, the statement 'p and q', will be represented by $p \cdot q$, and, if n = 5, the résumé p is represented by $p = p_1 \bullet (p_2 \bullet (p_3 \bullet (p_4 \bullet p_5)))$. Analogously, the symbol (+) will be taken for representing the linguistic disjunction 'or'; 'p or q' is represented by p + q; where the disjunction exclusive (either/or), the adopted symbol will be (∆), and 'either p or q' represented by $p \,\Delta\, q$, and understood as $(p + q) \cdot (p \cdot q)$', with (') denoting 'not'.

It should be recalled that, in general, the résumé p depends on the previous ordering of the premises; such ordering does correspond to (semantic) external criteria, concerning the meaning of the premises, and that those who reasons should capture along with the process of obtaining the previous information furnished by the premises. The model only can start once p is decided; without premises and its previous ordering no reasoning seems to be possible, from nothing only nothing can be drawn. The limit case is with $P = \{p_1\}$ and $p = p_1$; the following ones are $P = \{p_1, p_2\}$ with $p = p_1 \bullet p_2$, and $P = \{p_1, p_2, p_3\}$ with $p = p_1 \bullet (p_2 \bullet p_3)$, etc.

The main hypothesis to built up this model consists in (naïvely) supposing the existence of a, let's say, 'natural relationship of inference' among statements associating the premises with the conclusions, and symbolized by (<); then, the statement 'q is a conclusion of p', will be symbolically represented by p < q, shortening the linguistic conditional statement 'if p, then q'. Of course, this hypothesis leaves room to ask for which types of conclusions can exist. The only supposed property of the relation <, is the reflexive one: for any statement p, it is p < p, assuring that the relation < is not empty; there is, in the graph consisting in the statements and the relation <, and at least, a loop at each statement, that is, an arc starting and ending at it, and substituting <. It is adopted a naïve point of view on which the relation < is not directly defined, but naïvely supposed to correspond with the, let's say, observed operation of drawing conclusions from some previous information. Behind this, it should be supposed that p

and p < q actually indicate that both ‘p’ and ‘q is a conclusion of p’ are being asserted, by also taking ‘asserted’ in a naïve form; ‘p is asserted’ could be symbolically written by p!, but to avoid excessive symbolism it will be shortened by p unless the clarification could be judged necessary, and no concept of ‘true’, or ‘false’ is here considered. It is in the hands of those participating in the reasoning (just one, or several) to either accept, or not, that it is p! , (p<q)!, etc.

Notice that the relation < is not, in general, a linear one; almost always there are pairs of statements p and q, for which it is neither p < q, nor q < p; such statements are inferentially non comparable, or isolated. A typical example of isolated statements is given by p = Joan is a Lawyer, and q = San Francisco is a nice city. When a set of statements are enchained under <, it is said that they conform an inferential path, and, if for instance it were p < q, q < r, and r < s, it will be shortened by p < q < r < s, without presupposing neither p < r, nor q < s, nor p < s, except if it can be supposed that the ordered quadruplet (p, q, r, s) is transitive, that is, if p < q, and q < r, then p < r, for the triplet (p, q, r), and p < q, and q < s, then p < s, for the triplet (p, q, s), etc.

Of course, as more laws or properties could be attributed to the symbols ·, +, and <, more theoretical results can be reached; but before adding laws to these concepts, it should be experimentally checked if they actually hold in the context where natural language is employed for currently expressing ordinary reasoning. It is sane to be refractory on adding laws to the symbols; for instance, in classical logic it is presupposed that conjunction is commutative but, in natural language descriptions, it cannot be always supposed since almost always time intervenes (for instance, ‘she cried and entered’ vs. ‘she entered and cried’); it cannot be universally accepted that it always holds p • q = q • p. Often the linguistic and is sequential; natural language and commonsense reasoning are dynamic, but not static. For instance, in some cases formal deduction can be developed step by step and by means of inferential paths in which each step is actually attainable by using the known rule of ‘Modus Ponens’

$$(q < p)! \ \& \ p! : q! ,$$

but many times in commonsense reasoning what is sometimes the case is [(q <p)! & p*! : q*], where statements p* and q* are not exactly p and q, but close, or similar, to them as it is, for instance, with p = far away from my office, and either p* = very far away from my office, or p* = more or less far away from my office [28].

The negation of a statement p is another statement p’ just submitted to verify: If p < q, then q’ < p’. That is, negation can be seen as a mapping changing statements into statements that reverses the inferential relation. It

is obvious that there are several types of negations; besides the possibility that p and (p')' are isolated, there are the following:

- Intuitionistic negations, if (p')' < p

- Weak negations, if p < (p')'

- Strong negations, those that are both intuitionistic and weak; that is if (p')' < p, and p < (p')'.

Of course, the negation could not always follow one of the former laws but, in the discourse, it will depend on the particular statement p; that for different statements some of the former situations can hold.

When a < b and b < a, it is said that statements a and b are 'inferentially equivalent', and it is represented by a ≈ b. Hence, shortening (p')' by p", strong negations are characterized by p ≈ p". When it is neither p < q, nor q < p, it has been said that both statements are inferentially isolated, and, of course, it is also possible that in some cases p and p" can be isolated.

When it is a < b', not b is a conclusion of a, b is inferentially contradictory with a, and when a < a', such a is inferentially self-contradictory; no self-contradictory element can be neither accepted as a premise, nor as a résumé, nor as the conclusion of any inference. Self-contradiction should be avoided as something not at all acceptable; even it cannot be accepted the existence of pairs of contradictory premises. Contradiction is the worst (natural) 'sin' in acquiring the previous information for doing any process of inference; its apparition among the premises is not at all admissible, is a red line for continuing the process. Nobody accepts a statement from which its negation can be inferred, nor two premises from one of which the negation of the other is inferred.

Remarks

The following remarks (a) to (f) concern the singular role self-contradiction and tautology play in inference.

a) If q < p, and p < p', it is also p' < q' and provided the quadruplet (q, p, p', q') is transitive, it follows q < q'. Hence, and under transitivity, self-contradictions only can be inferred from self-contradictions.

b) Since p' < p implies p' < (p')', it means that p' is self-contradictory, and p is called a 'tautology'. Notice that the reciprocal, p' < (p')' => p' < p, cannot be proven without supposition on the negation; for instance, if it is intuitionistic or strong, equivalence is verified. Under a intuitionistic, or a strong negation, 'p is self-contradictory' and 'p' is a tautology' are equivalent.

If $p < q$ and $p' < p$, it is $q' < p'$, and provided (q', p', p, q) is transitive, it follows $q' < q$. Hence, from tautologies only tautologies can be inferred.

c) If p is a self-contradiction, nothing structural can be said in general on the existence of a non self-contradictory q such that $p < q$. But, for instance, in [0, 10], with $<$ equal to the order $\leq$ of numbers, and the strong negation $p' = 10 - p$, it is $p \leq p' \Leftrightarrow p \leq 5$, and any q such that $5 < q$ is non self-contradictory and verifies $p \leq q$. It seems actually possible, and even reasonable, the possibility of inferring non self-contradictions from self-contradictions; on the contrary, it could be said that the self-contradiction is forwards-inferentially isolated.

d) If p is a tautology, nothing structural can be said in general on the existence of a non tautology q such that $q < p$. But, for instance, in the former interval it is $p' < p \Leftrightarrow 5 \leq p$, and any q such that $q < 5$ is not a tautology and verifies $q \leq p$. It seems actually possible and even reasonable, the possibility of inferring tautologies from a non tautology; on the contrary it could be said that the tautology is backwards-inferentially isolated.

e) Provided the statements are in an Ortho-lattice (or, in particular, in a Boolean Algebra), it is $p \leq p' \Leftrightarrow p = 0$, and the only self-contradiction is 0. It is $p' \leq p \Leftrightarrow p = 1$, and the only tautology is 1. Hence, for all q is $0 \leq q \leq 1$. If the statements are in a De Morgan Algebra, there are the self-contradiction 0 and the tautology 1, that both are unique among its Boolean elements, but, as it is shown in [0, 10], there should exist more self-contradictions and tautologies; on the contrary all its elements will be Boolean. For instance, it is $p \cdot p' \leq (p \cdot p')'$, and $(p + p')' \leq p + p'$, for all p in a De Morgan Algebra [23].

f) The existence of too many self-contradictions and tautologies (whose 'number' cannot be established, in general, as being always equal), depends on the structure of the set of considered statements, that should be strong enough for reducing those numbers to just one. For instance, if in a universe of discourse X are considered all the words that are either imprecise, or precise, then and instead of the set of words, it can be considered the set of all fuzzy sets in X specifying the words, that is, a subset of $[0, 1]^X$, where it is $\mu \leq \mu' \Leftrightarrow \mu \leq N \circ \mu$, with N a strong negation function given by $N(a) = \Delta^{-1}(1 - \Delta(a))$, for all a in [0, 1], and some order's automorphism Δ of [0, 1], with which it is $\mu \leq \mu' \Leftrightarrow \mu \leq \mu_r$, with $r = \Delta^{-1}(1/2)$, that is, μ is below the function constantly equal to r. Analogously, it is $\mu' \leq \mu \Leftrightarrow \mu_r \leq \mu$, with the former r. Hence, all functions below the level r are self-contradictory, and all of them greater than such level are tautologies. There are not only many self-contradictions and tautologies, and many others that are neither the first, nor the second, but among each type there are functions that are not comparable, or isolated, under the ordering $\leq$. It can be believed that in (ordinary) discourses with a large number of imprecise words there can be several self-contradictions, and several tautologies; rarely, these numbers will be reduced to one.$\Diamond$

3. Once given a set P of premises with résumé p, and a negation is selected, it will be said that q *refutes* or is a *refutation* of P, if p < q', that is, q is contradictory with p. What is not a refutation is a *conjecture* of P, that is, conjectures are those statements verifying p </ q', those that are not refuting p. Obviously, once P and p are given, there are no more statements than conjectures and refutations. The expression p </ q' means either q' < p, or that p and q' are inferentially isolated; it only excludes p < q'. Since two statements a and b whatsoever are either isolated, or verify a < b, or b < a, thus, given P and p, there are only three classes of conjectures q such that p </ q':

1) Those q such that p < q, called *consequences*

2) Those q such that q < p, called *hypotheses*

3) Those q such that are isolated from p, called *speculations*.

Hence, conjectures are classified in consequences, hypotheses, and speculations [11, 26].

Concerning consequences, it should be noticed that provided the résumé p were not self-contradictory (not p < p'), and the ordered triplet (p, q', p') is transitive, 'a fortiori' they are conjectures. To prove it, notice that on the contrary, since it should hold p < q', from p < q follows q' < p', and from the triplet's transitivity do follow p < p', that is absurd. Hence, all statement q such that p < q is, under the above suppositions, a conjecture. Under general enough conditions, consequences are a particular case of conjectures; in a sense to be further clarified, in this model deduction appears as a particular case of conjecturing, or guessing. In the same vein, non self-contradictory hypotheses are also conjectures: If q < p and it is p < q', just provided the triplet (q, p, q') is transitive, it follows the absurd q < q'. Under transitivity, both p < q and q < p imply p </q'. For what refers to speculations of P, they are immediately classified in just the two following classes,

- Those q such that q' < p, with p and q isolated (for short, p NC q, with NC = Not comparable under <, or isolated)
- Those q such that p NC q' and p NC q,

The first class is called that of 'hypothetic speculations', since the negation q' of the speculation q can be seen as a hypothesis for P. The second class is called that of 'creative speculations' since no inference is possible between p and q, nor between p and q': Both q and q' are inferentially isolated from the résumé p. Creative speculations can be seen in the roots of 'imagination', 'illumination', etc., they help to produce the famous Archimedes' *Eureka!* (I found it!).

For what concerns to refutations q of p (p < q'), that is, those statements that are contradictory with the premises résumé, and that cannot be conjectures, it should be pointed out that no one of them can be a consequence provided the triplet (p, q', p') is transitive, since if p < q, then q' < p', and with p < q' it follows p < p', that is absurd. Analogously, and provided the triplet (q, p, q') is transitive, there is no any non-self-contradictory refutation q that can be a hypothesis, since if it were q < p, it will follow q < q'. Obviously, and even if it is p NC q, they cannot be speculations since it is p < q'.

To know if there are actually refutations or conjectures that, lacking transitivity, can simultaneously belong to several of the before reached classes, is a currently open question for which finding some examples could be interesting.

If, for the same P, the existence of two contradictory hypotheses or speculations is not at all surprising, what there cannot exist are pairs of contradictory consequences. Provided p < q and p < r, if it were q < r', and it can be supposed transitive the triplet (p, q, r'), it follows p < r', and it is reached the absurd that r is also a refutation of P, when it is known that no refutation that is a consequence can exist. Obviously the before deduction holds with r = q, hence there cannot be self-contradictory consequences. Since no similar proof is available for hypotheses and speculations, these conjectures should be left aside if they are self-contradictory; that is, not to be considered as proper conjectures.

Notice that all these results are obtained under some hypotheses concerning triplet's transitivity; hence, all of them are safely valid whenever it can be supposed that < is fully transitive, that all the involved triplets or quadruplets are transitive, as it is in the cases of classical, quantum and fuzzy logic. Notwithstanding, its validity without transitivity is still an open question.

Remarks

What is not the case is that both consequences and hypotheses should be always comparable. For instance, a very simple example concerning the existence of non comparable, or isolated hypotheses, is the following. There are four possible hypotheses, for p = The child is crying: q_1 = S/he is not allowed to watch TV; q_2 = S/he was harshly spoken; q_3 = S/he suffered a physical pain; q_4 = Watching TV s/he did look at a violent scene; ... All of them are inferentially isolated from p and even q_1 and q_4 are contradictory.

For what relates to consequences, if they cannot be contradictory they are often isolated; for instance, with p = Mr. X failed down by the stairs, we can consider q_1 = He broke a leg, and q_2 = He broke the neck and die, that are inferentially isolated consequences.◊

4. Notice that it is equivalent to say that q is a conjecture, or refutation of P, or to say it is so of the singleton {p}. This is why p is seen as the résumé of P; it contains, under the current definitions, all the needed inferential information. What happens when information grows, that is, when new premises are added to P? For analyzing such 'character of monotony' of conjectures and refutations, it is sufficient to agree in the non surprising laws,

$$p \cdot q < p, \text{ and } p \cdot q < q,$$

under which if a new premise p_0 is added to P, that is, we pass from P to $P^* = \{p_0, p_1, \dots, p_n\}$, then the new résumé $p^* = p_0 \cdot [p_1 \cdot \dots] = p_0 \cdot p$, verifies $p^* < p$. Hence,

- If q is a consequence of P ($p < q$), and the triplet (p^*, p, q) is transitive, it is also a consequence of P*, since from $p < q$, and $p^* < p$, it follows $p^* < q$. That is, under transitivity conditions, when the number of premises increases, the number of consequences cannot decrease; this is known by naming it the monotonic property. Consequences are monotonic: more information cannot lead to less conclusions.
- If q is a hypothesis for P*, $q < p^*$, and the triplet (q, p^*, p) is transitive, since $p^* < p$, it follows $q < p$, and q is also a hypothesis for P: when premises increase, hypotheses cannot increase. Hypotheses are anti-monotonic: more information cannot lead to more explanations.
- If q is a refutation of P, $p < q'$, since $p^* < p$, if the triplet (p^*, p, q') is transitive, it follows $p^* < q'$, and q is a refutation of P*: Refutations are monotonic.
- By its part and with analogous transitivity conditions, conjectures at large are anti-monotonic: If $p^* \not< q'$, since $p^* < p$, if it were $p < q'$, it will follow the absurd $p^* < q'$.
- Finally, and concerning speculations, the subclass of hypothetic ones is anti-monotonic: both $q' < p^*$, and $p^* < p$, imply $q' < p$. But, for the subclass of creative speculations neither monotony, nor anti-monotony hold, there is no a fix law of growing for them; it is, in general, unknown if more information will lead to more or less creative speculations. For what corresponds to the impossibility of being reached through inferential paths, and to its previously un-decidable way of growing, creative speculations can be seen as 'wild' conjectures since they cannot be deductively captured, cannot be deductively domesticated. Since there don't exist ways of reaching

a creative speculation by any inferential path, they are included in the true *inductive* part of reasoning.

In sum, reasoning just consists in the natural processes for conjecturing and refuting, and conjecturing in those of deducing (obtaining consequences), abducing (obtaining hypotheses or explanations), and speculating that, at its turn, consists in inducing (obtaining creative speculations), and in the backwards deductive processes for obtaining the negation of hypothetical speculations. Notice that, under transitive assumptions, consequences can be reached by forwards deductive paths from the résumé, hypotheses by backwards deductive paths from the résumé, and hypothetical speculations q by backwards deductive paths from the résumé up to its negation q' that, at its turn and depending on the type of negation, can allow to reach q by either double negation and backwards deduction (if it is $q < (q')'$), by double negation and forwards deduction (if it is $(q')' < q$), or simply by double negation if $q \approx (q')'$. The only case in which q' does not allow to reach q is when it is isolated from (q')', and then the process is also inductive.

In sum, creatively speculating and some cases of hypothetic speculating, are the only form of totally non-deductive reasoning; they are properly obtained by inducing. The only way for reaching such speculations is heuristic, inductive, and only the other types of reasoning can be subjected to algorithmic methods. Notice that refutations can be reached by forwards deduction from the résumé p up to the negation q' of the refutation q; from it, and depending on the negation's type, it is sometimes possible arriving at the refutation; newly, if q and q'' are isolated, q is only reachable by means of heuristics, and the process is also inductive. Hence, induction just refers to finding creative speculations and, in some cases, hypothetic speculations and refutations. Only these cases are deductively wild.

It should be pointed out that, nevertheless, speculations can be, in some cases, reached by a sequence of forwards (backwards) and backwards (forwards) deductive paths requiring to have information on both p and q; a very simple example in a finite Boolean algebra is in order for it. Let it be a Boolean algebra with five atoms a, b, c, d, and e, $p = a + b + c$, and $q = b + d$. Even if $p \cdot q = b$, p and q are isolated, and since $q' = a + c + e$, also p and q' are isolated; hence q is a creative speculation from p. Since $b \leq p$, and $b \leq q$, it is possible to go backwards from p to b, and forwards from b to q; to reach q from p requires to know that b is a part of p and a part of q, to have some information on what constitutes p and q that is, in practical cases, a semantic subject.

5. Nevertheless, it yet lacks to take into account a conspicuous form of ordinary reasoning, analogy. A first question on it is if conclusions reached

by analogy are actually different from conjectures and refutations or analogy is, simply, a powerful natural way for obtaining conjectures and refutations. To begin with such question, it will be established a symbolic representation of what analogy is, that is, a specification of analogy.

Let's suppose that some inferential processes on a particular subject were conducted in a universe X, and that Y is another universe in which also reasoning on its objects is done. Let's also suppose that the inferences done in X are stored in a memory (ours, for instance). An inferential analogy can be represented by a mapping A: X Y, such that if p < q, then A (p) < A (q). Notice that, provided it is p ≈ q, it follows A (p) ≈ A (q); inferential equivalence is preserved by analogy: If it not the case with some mapping A, it is not an inferential analogy. Analogy is, under such definition, nothing else than a morphism of inference. Provided A is one-to-one, it can be just considered as a bijection between X and its image A(X) in Y, and since then X and A(X) have the same cardinal, the study can be reduced to one-to-one and onto mappings X X.

For each x in X only one of the three following conditions can hold: 1) x < A(x); 2) A(x) < x; 3) x and A(x) are inferentially isolated. In the first case, what is reached by analogy is a consequence of any set of premises whose résumé is x; in the second, is a hypothesis of any such set of premises, and in the third is a speculation. For those elements also verifying x < A(x)', what will be obtained are refutations, and conjectures for those such that x </ A(x)'. Hence, under the before representation of analogy as an inferential morphism, only conjectures and refutations can be reached, and, in particular, consequences, hypotheses, and speculations. Analogy does not facilitate a new kind of conclusions.

Provided analogy were not defined as a mapping but as a correspondence, or relation, the result just obtained can be also retained by selecting, for each x in X, just one element in Y among the several the relation associates to it; of course, those x without a single correspondent in Y are out of consideration and can be forgot. Hence, if defining A as a correspondence, or subset in X x Y, instead as a mapping between X and Y, seems closer to what analogy is actually, nevertheless and when reasoning by analogy, it is usually taken just an 'analogous' object A(x) to x, and not a collection of them, and the bijective hypothesis can be kept by restricting to the pairs (x, A(x)). Consequently, reasoning by analogy does not actually conduct to a different type of conclusions than conjectures and refutations [11]. Anyway, the great number of possibilities existing for selecting either a correspondence, or a mapping A, from what is stored in a memory, and into new scenarios, allows an enormous capability for reasoning under experience. In conclusion, what analogy offers is a powerful tool for doing reasoning in the basis of past experiences.

Anyway, it should be noticed that between x and A(x) it can exist many, and perhaps important, differences; x and A(x) can keep analogy under some aspects, but be very different in others. For instance, from the point of view of its spherical character, apples and oranges can be seen as very similar, but from many other points of view (color, taste, smell, etc.) they show important dissimilarities. Hence, in any system for doing reasoning by analogy, the degree of analogy between each pair x and A (x) with respect to as much aspects of both similarity and dissimilarity that could be taken into account, should be controlled doing it through either quantitative indices of analogy [29], or and at least, qualitative considerations. A serious danger of analogical reasoning lies in the dissimilarities shown on some attributes of the corresponding objects; an as complete as possible list of these attributes should be always considered.

By the way, it should be observed that Modus Ponens, $p \cdot (p < q) : q$, is not sure to hold after applying the analogy A, that is, $A(p) \cdot (A(p) < A(q))$. It is unless it holds $A(p \cdot (p < q)) = A(p) \cdot A(p<q) = A(p) \cdot (A(p) < A(q)$. For it, A is required to verify some transformation laws like it is $A(x \cdot y) = A(x) \cdot A(y)$, not actually easy to check. Hence, it should be carefully considered if a valid deduction in X is actually preserved in Y by analogy.

6. The simple symbolic model presented allows formalizing two aspects of some relevance.

The first of them concerns how hypotheses can be falsified, something Karl Popper did introduce in his philosophy of science as a basic demarcation for the development of the scientific processes of search [3, 30]. Let's suppose that it is suspected h is a hypothesis for a corpus of evidence with résumé p; it means the supposition $h < p$.

a) Provided the triplet (h, p, q) is transitive, from $p < q$ follows $h < q$: any consequence of P is also a consequence of {h}. Hence, to falsify h (stating it is not actually a hypothesis), it suffices to find a consequence of P that is not one of {h}; a single one suffices. This is the kind of falsification corresponding to deductive reasoning; if not only, it is typically followed in the formal proofs of mathematics by 'reductio ad absurdum'.

b) Provided what is known is that q is a conjecture of {h} ($h \not< q'$), q is also a conjecture of P: were it not so, that is, were it $p < q'$, from $h < p$ and newly supposing transitive the triplet (h, p, q'), it follows the absurd $h < q'$. Hence, it is $h \not< q'$, and q is a conjecture of {h}. Consequently, to falsify h it suffices to find a single one of its conjectures not being one of P. This is the kind of falsification corresponding to both experimental and soft sciences, as well as to philosophy, where usually conclusions are but conjectures.

The second aspect, corresponds to a nice benefit speculating offers to reasoning. Let it be s a speculation of P.

a) The statement, ‘p and s’ (p · s), if it is not self-contradictory, is a hypothesis for P (explains P), since it is p · s < p.

By just agreeing in the also non surprising laws p < p + q, and q < p + a, it follows,

b) The statement ‘p o q’ (p + q) is a consequence of P, since it is p < p + q.

Notice that the same result cannot be generally stated for the strict disjunction ∆, since what implies p ∆ q = (p + q) • (p • q)’ is just p ∆ q < p + q, but without usually being p < p ∆ q that, for instance, in the Boolean case, implies p ≤ q’: In such case is p ∆ q = p · q’ + p’ · q, and thus p ≤ p ∆ q implies p · (p ∆ q) = p, or p · q’ = p, equivalent to p ≤ q’. It will be necessary that q were a refutation of P.

These results show the usefulness of speculating when either a consequence, or a hypothesis, cannot be found directly. It suffices to find a speculation of P, its conjunction with the résumé is a hypothesis, and its disjunction is a consequence [11, 32]. It seems to remember what sometimes is actually done by people when searching either for what logically follows, or for an explanation.

Let us say, again, that a set of premises originates sets of refutations, conjectures and, in particular, of consequences, hypotheses, and speculations. Usually, all these conclusions are not a single one, even if the sets of hypotheses or speculations can be empty the second, and reduced to p the first. At each case, but especially in that of searching for a hypothesis, finding the best among several suitable ones is very important. In the same line, there are consequences appearing as obvious, and others that seem to be significant and that, for instance, mathematicians denote by theorems, propositions, corollaries, etc. Finally, a good speculation is one allowing either a good enough hypothesis, or a significant consequence.

7. Let us return for a while to probability by just considering the case of throwing a dice, whose possible elementary results are subsets in X = {1, 2, 3, 4, 5, 6}, symbolizing the points that can be obtained. For instance, to bet in ‘an even number of points will be obtained’ refers to the subset {2, 4, 6} of X; the events to which probability can be applied are the subsets of X, and those with probability 0 and 1 are, respectively, the empty Ø, and the total X. Notice that, interpreting < as the set inclusion ⊆, consequences are those subsets q such that p = X ⊆ q, that is, there is only the single consequence q = X. Hypotheses are those h such that h ⊆ X, and refutations are those

subsets r such that $r \subseteq X^c = \emptyset$, that is, the only refutation is the empty set. Obviously, in this case there are not speculations. Hence, events are just hypotheses, and the only that are not allowed to bet in are the résumé X (it is the sure event!), and the refutation Ø (it is impossible). This example also shows why it is often considered that hypotheses not only are those h verifying $h < p$, but also $h \neq 0$ (the minimum, if it exists).

Part Three

IV Further Comments

1. A first comment concerns the essential concept of conditional, that is, the relationship of linguistic statements 'if p, then q' (p < q) with inference. Almost all we know, if not all we know, are conditionals; for instance, any theorem on natural numbers, are of the type 'If Peano's axioms are accepted, then so and so', a common sense statement like 'the Bank will refuse the check' is actually a knowledge of the type 'If my information on the Bank behavior and the economic situation of the accountant is correct, then the Bank will refuse its check', and so forth. A modification on what is known on the antecedent p can invalidate the consequent q; for instance, provided what I know on the economic situation of the accountant did change in the last days, it is perfectly possible that the Bank will not refuse its check. Hence, for what concerns conditional, there are two basic questions; the first is the conditional's representation that depends on the available contextual information on it, and the second is its character in making inference.

1-a. The representation of a conditional means to identify p < q with a linguistic expression involving the statements p and q, as it is done, for instance, in classical Boolean logic by identifying pq with p' + q = 1, or in Quantum Orthomodular logic with pq $\Leftrightarrow$ p' + p · q = 1, or q + p' · q' = 1, or in some cases in commonsense reasoning in which p < q is understood as (p · q)! (affirming the conjunctive conditional form). It should be noticed that all the former formulae are of the 'material' type, that is, they are just algebraic expressions equivalent to the tautology 1 (the maximum in the lattice, translating total assertion) in which the parts of the first member are just joined by the connectives and (·), or (+), and not ('). Its most general Boolean form is p -> q = α · p · q + β · p' · q + ε · p' · q', with α, β, and ε equal to 0 or to 1; for instance with α = β = ε =1, it is p -> q = p · q + + p' · q + p' · q' that, in a Boolean algebra, coincides with p' + q. In such cases it is said that the representation is truth-functional, but this cannot be always supposed to be the case, as it happens, for instance, with the so-called contrafactual conditionals, those whose antecedent is not reflecting what actually did happen. For instance, in Boolean algebras, and if the conditionals are submitted to satisfy the Modus Ponens inequality p · (pq) ≤ q $\Leftrightarrow$ pq ≤ p' + q, it is clear that any expression less or equal than the classical material conditional pq = p' + q, is a conditional; among them:

- p q = p' + q; p' · q + p · q; p · q; q, ...,
- p q = q, provided p = 0; p q = p · q, provided p ≠ 0,
- p q = q, provided nothing is known on p; p q = p · q otherwise.

Notice that all these conditionals partly belong to the material type, and that all of them verify, in Boolean algebras, the Modus Ponens inequality because of all verify $p \cdot (p\ q) = p \cdot q \leq q$, its left member $p \cdot (pq)$ coincides with the conjunction of antecedent and consequent, the résumé of the premises {p, p q}. What is clear is that even in a strong lattice structure like a Boolean algebra, corresponding to the texture of mathematical language, there are many expressions able to represent a conditional [33], and that in the softer structure of an Orthomodular lattice, corresponding to the quantum physics language, there are also many, like they are for instance the so-called Sasaki $(p' + p \cdot q)$, and Dishkant $(q + p' \cdot q')$ hooks. Notice that these last conditional reduce, when the distributive law holds, to the classical material conditional: $p' + p \cdot q = (p' + p) \cdot (p' + q) = 1 \cdot (p' + q) = p' + q$, and $q + p' \cdot q' = (q + p') \cdot (q + q') = q + p' = p' + q$. Notice also that if in Boolean algebras there exists the maximum conditional $p' + q$, in Orthomodular lattices there are conditionals that are only maximal, like the Sasaki and the Dishkant that, of course, are not order-comparable since it is neither $p' + p \cdot q \leq q' + p' \cdot q'$, nor $q' + p' \cdot q' \leq p' + p \cdot q$, they are isolated conditionals.

Which is the case in the open textured natural language is an important but open question, for whose study it seems necessary to adopt an experimental way, with a previous analysis able to suggest how, where, and when, some equivalent expressions can be identified with $p < q$. It is, for instance, known that in some cases it is $(p < q)! \approx (p' + p \cdot q)!$, but in others $(p < q)! \approx (p \cdot q)!$ It is an open problem.

1-b. The usefulness of conditionals in inference is directly related with the relation between the information contained in the set of premises {p, $p < q$} and that contained in the singleton {q}. That is, between the résumé $p \cdot (p < q)$ and the conclusion q, with the supposition that neither the résumé is self-contradictory, nor p and $p < q$ are either self-contradictory, or contradictory. Notice, for instance, that in no algebraic case (either Boolean, Orthomodular, or any other) it is acceptable $p = 0$, that the résumé is 0, and it does not depend on the consequent q; in this situation the résumé is self-contradictory. Conditionals $p < q$ in which it is not possible to state p (p!), $p < q$ ($(p<q)$!), and the résumé ($(p \cdot (p < q)$!), are not considered; the usual saying 'From false it follows everything', is also not accepted by the authors, from false it cannot follow anything. For what concerns the usefulness of conditionals in inference, let us only consider the following four possibilities: 1) $p \cdot (p < q) < q'$; 2) $p \cdot (p < q) < q$; 3) $q < p \cdot (p < q)$; 4) $p \cdot (p < q) \not< q'$, respectively saying that q falsifies the résumé (1), q is a consequence of the résumé (2), q is a hypothesis for the résumé (3), and q is a conjecture of the résumé (4).

Whenever q is a conjecture, but it is neither a consequence, nor a hypothesis of the résumé, the conditional will be said a conjectural one, and it will be said a deductive one if q is either a consequence, or a refutation of the résumé; in the last first case it is a 'purely deductive' conditional. Of course, all the conditionals before presented in Boolean algebras are purely deductive and not refuting ones. For example, if it were $p \cdot (p' + q) \leq q'$ (q refutes the résumé), with $q = 1$ it will follow $p \cdot (p' + 1) = p \cdot 1 = p \leq 1' = 0$, and hence $p = 0$, with which it is also $p \cdot (p' + q) = 0$; thus both the antecedent and the résumé should be self-contradictory, and nothing can be concluded.

Concerning the conjectural or non deductive conditionals, the only resting in the former classification are those that are speculative, and that at its turn can be classified as those verifying $q' < p \cdot (p < q)$, and those verifying $p \cdot (p < q)$ NC q', that is, those such that the résumé and q' are inferentially isolated in addition to the isolation between the résumé $p \cdot (p < q)$ and the consequent q.

For instance and in the Boolean case, where the material conditional hypothetical (case 3), from $q \leq p \cdot (p' + q) = p \cdot q$, it will follow $p \cdot q = q \Leftrightarrow q \leq p$, that is equivalent with q being a hypothesis for p. The same conclusion follows with, for instance, all conditionals expressible in a form such that the résumé is $p \cdot q$. Obviously, to obtain conditionals able to explain the résumé, the Boolean inequality $q \leq p \cdot (p \ q) \Leftrightarrow p' + q \leq p' + (pq)$, should be solved or, at least, some of its solutions found; since $q \leq p \ q$ implies the former inequality, it suffices that p q verifies $q \leq p \ q$, and hence $p \ q = q$ is a minimal hypothetical conditional that, and notwithstanding, is also purely deductive. Analogously, to find the solution to case (1), that is, conditionals able to refute the résumé, the inequality $p \cdot (pq) \leq q'$ should be solved; it implies $p \ q \leq p' + q'$, and reciprocally since from this last inequality follows $p \cdot (pq) \leq p \cdot q' \leq q'$. Hence, all the solutions should verify $p \ q \leq (p \cdot q)'$, the negation of the conjunctive conditional is the greatest one that is able to refute the résumé.

For what, also in Boolean algebras, concerns case (2), the purely deductive conditionals, since the inequality $p \cdot (p \ q) \leq q$ is equivalent to $p \ q \leq p' + q$, the classical material conditional is the greatest one under which q is a consequence of the résumé. A typical commonsense example of a conjectural and not deductive conditional is given by p = 'Mr. X buys a lottery ticket', q = 'Mr. X will win the lottery's first award', in which it is clear that it is not $p \cdot (p < q) < q$, but that it can be stated $p \cdot (p< q) \not< q$'. If instead of that, and provided it were q = 'Mr. X won the first lottery's award', and p = 'Mr. X bought a lottery ticket', then what holds is $q < p \cdot (p < q)$.

Obviously, and also in Boolean algebras, finding algebraic expressions of the speculative conditionals with the negation of the consequent isolated from the résumé (creative speculations), is not possible by solving any

functional inequality; on the contrary, the inequality $q' \leq p \cdot (p \ q)$ implying $q' \leq p \ q$, could offer some solution for the speculative conditionals whose résumé is not isolated from the negation of its consequent.

Finally and to summarize, up to now conditionals only have been considered from the point of view of deductive reasoning, but for the analysis of commonsense reasoning it still lacks to study conditionals from the point of view of conjecturing and refuting. In particular, it lacks to be known both their material and non-material expressions, as well as functionally and not functionally expressible forms of them; all that within the relevant goal of understanding how conditionals act in language, reasoning, and knowledge, and for using them towards mechanizing commonsense reasoning. In the main, all this is an open problem that contains that of separating the diverse types of conditionals. It should be stressed that a correct management of conditionals is a proof of reasoning's maturity; for instance, for children it takes some time to correctly manage conditionals.

2. It still deserves to be commented why no reference to the concept of Truth has been made, and only linguistic assertion is taken into account; why the typically philosophical expressions 'statement p is true', or 'statement p is false', never appeared among this paper's arguments. The reason comes from seeing at Truth as a non universal, but often metaphysical, concept depending on what can be understood by true, and its antonym false not always coincidental with its negation not-true; something that is nothing else than the meaning of the predicate 'true' that, always applied to statements is, like all predicates, context-dependent and purpose-driven. Nevertheless, a glance on the meaning of the predicate true in a universe of statements, and some comments on its inferential use, seems to be something necessary for, at least, clarifying its eventual use.

2-a. Let it be X a universe consisting in statements p, q, r, etc., and T the predicate true with which new statements 'p is T' are considered. Once it can be recognized when it is 'p is less T than q', it can be known the graph $(X, <_T)$, or qualitative meaning of T in X. With this graph a measure of the extent up to which p is true can be specified; it is any mapping $t: X \to [0, 1]$ such that: 1) $p <_T q \Rightarrow t(p) \leq t(q)$; 2) If r is a maximal in the graph, $t(r) = 1$; 3) If s is a minimal, $t(s) = 0$. Of course, if T is irregular, the graph is reduced to $(X, =_T)$, only the values 0 and 1 are possible for the single t that can be defined, the totally true statements are those in $t^{-1}(1)$, and the totally false are those in $t^{-1}(0)$. In general, and unless T is irregular in X, predicate T has as many meanings as triplets $(X, <_T, t)$ can be defined; in language the qualification of true does not always mean the same, and it should be noticed that, in language, T is usually a regular predicate.

2-b. How all that can be linked with inference, that is, what can be said on the true character of an inference once $p < q$ is understood as a statement? Which property, or properties, t should enjoy for saying something, given the truth degree $t(p)$ of the antecedent, on the truth value $t(q)$ of the consequent q?

Provided t is such that $p < q \Rightarrow t(p) \leq t(q)$, then the degree of true of the conclusion is greater or equal than that of the antecedent. Analogously, if q is a refutation of p, $p < q'$, it is $t(p) \leq t(q')$, and if q is a hypothesis of p, $q < p$, it is $t(q) \leq t(p)$, that is, the degree of true of the negation of a refutation is always equal or greater than that of the antecedent, but the degree of true of a hypothesis is always equal or less than that of the antecedent. Concerning the other conjectures, $p \nless q'$ does not permit to compare immediately $t(p)$ and $t(q')$, unless it is $q' < p$ in which case follows $t(q') \leq t(p)$. Only for hypothetic speculations the degree of its negation is as much the degree of the résumé, and, hence, is false if the résumé is also false. If T is irregular, and q is a consequence of a true résumé $t(p) = 1$, it is $t(q) = 1$; that is, in the irregular case, from true premises follow true consequences; if q is a refutation, what follows is $t(q') = 1$, that is, the negation of the refutation is true, and if the negation of the refutation is false, the résumé is also false. In the case of irregular hypothetic speculations, from a true résumé it can follow either a true or a false negation of it, but from a false résumé only false negations of a refutation can follow. The same happens with irregular hypotheses, for a true résumé there can be obtained either true or false hypotheses, but for a false résumé only false hypotheses can exist. Notice that provided it were $t(q') = 1 \Leftrightarrow t(q) = 0$, like it happens if $t(q') = 1 - t(p)$ as it is in the classical case, it can be stated the true or the false character of refutations and hypothetic speculations.

Opposite results are obtained provided it were $p < q \Rightarrow t(q) \leq t(p)$. Anyway and from a deductive point of view, this will be a counter-intuitive law for the degree of true, since to reach consequences less true than the résumé, and hypotheses more true than the résumé, only can be seen as something actually bizarre. Thus, only those degrees of true such that $p < q \Rightarrow t(p) \leq t(q)$ can be accepted.

2-c. What it yet lacks to study is the possibility of having formulae for the degrees of $t(p \cdot q)$, $t(p + q)$, $t(p')$, and $t(p < q)$, depending on $t(p)$ and $t(q)$, like they exist in the classical irregular case in Boolean algebras. It is of interest, for instance, to bound the truth value of q from $p \cdot (p < q) < q$, where it is $t(p \cdot (p<q)) \leq t(q)$. Provided it were $t(p \cdot q) = f(t(p), t(q))$, and $t(p<q) = g(t(p), t(q))$, it will follow the inequality $f(t(p), g(t(p), t(q)) \leq t(q)$, and perhaps a bounding for $t(q)$ if, for instance, it were $b \leq f(a, b)$. If like in the classical case, f is minimum, $f(a, b) = \min(a, b)$, and $g(a, b) = \max(1-a, b)$,

from min (t (p), max (1-t (p), t (q)) ≤ t (q) ⇔ max (min (t (p), 1- t (p)), min (t (p),t (q))) ≤ t (q), it follows the low bounding min (t (p), 1-t (p)) ≤ t (q), equivalent to

$$t(p) \leq t(q), \text{ if } t(p) \leq \tfrac{1}{2}$$

and,

$$1\text{-}t(p) \leq t(q), \text{ if } t(p) > \tfrac{1}{2}.$$

Anyway, those expressions depend of the meaning, in the corresponding language's context, of the linguistic connectives *and, or, not,* and *if/then,* whose study is an almost open problem as it have been said formerly.

In the limit situation in which all the statements are irregular, and t can be taken as a Kolmogorov's probability, t = prob: Ω [0, 1], in a Boolean algebra Ω in the universe X, it appears an interesting case for what concerns the meaning of the conditional. As it is well known, and provided it is prob (p) > 0,

$$\text{prob}(q/p) = \text{prob}(p \cdot q) / \text{prob}(p),$$

the Bayes' formula, represents the probability of p < q, of q if p, a situation in which such corresponding linguistic conditional cannot be interpreted in Ω as the material form p' + q, since it gives not a probability like prob (q/p) does. The conditional, necessarily needs to be interpreted as 'p · q' for getting its probability and provided as it is in a Boolean algebra, that the conjunction (·) is commutative Notice that if prob is a probability in Ω, also prob (. / p) is a probability in the set of the traces (q · p) all its elements q have in p; that is, prob (. /p) is also a probability but in the restricted Boolean algebra constituted by the elements in $\Omega^* = \{q^* = q \cdot p; q \text{ in } \Omega\}$. In some sense, p represents a kind of diaphragm only allowing to consider what 'is inside p'; the antecedent is the new universe, hiding what is out of it, and where, for instance, the negation of q is not q' but $q\hat{} = q' \cdot p$. Thus, $p\hat{} + q^* = p' \cdot p + q \cdot p = 0 + q \cdot p = p \cdot q$; in Ω^*, inside the new and restricted universe p, the material conditional coincides with the conjunctive conditional. Notice that if two possible antecedents, p_1 and p_2 with empty or contradictory conjunction are possible, then it is

$$\text{prob}(q/ p_1 + p_2) = \text{prob}(q \cdot p_1 + q \cdot p_2) / [\text{prob}(p_1) + \text{prob}(p_2)],$$

since the distributive and associative laws hold in Boolean algebras, it is: $q \cdot (p_1 + p_2) = q \cdot p_1 + q \cdot p_2$, with $(q \cdot p_1) \cdot (q \cdot p_2) = q \cdot (p_1 \cdot p_2) = 0$, and being prob an additive function it follows $\text{prob}(q \cdot (p_1 + p_2)) = \text{prob}(q \cdot p_1) + \text{prob}(q \cdot p_2)$.

Last comments suggest a view on measuring conditionals in forms similar to t (p < q) = t (p · q) / t (p), perhaps with a different expression than the conjunctive for the conditional, and without t being a probability but just a general measure of T = true. Notice, for instance, that in the Bayes formula the conjunction (p · q) is supposed commutative since the statements/events are irregular/classical sets, but that, when the statements are regular/fuzzy sets, such commutative property cannot be generally supposed. The values t (p) and t (p · q) are called the *a priori* measures, and t (p < q) =: t (q/p), the *a posteriori* ones. Provided the conjunction is commutative, it is easy to find the value of t (q < p) in function of t (p < q), since: t (p < q). t (q) = t (p · q) = t (q · p) = t (q < p). t (p), or

$$t (q < p) = [t (q)/t (p)]. t (p < q),$$

that is the expression for truth of the inverted conditional where the old consequent passes to be the new antecedent.

In some cases, the function t (q /p) =: t (p · q) / t (p), is also a measure inside p, and not only a degree concerning the validity of p < q. In fact, it is: 1) $q_1 < q_2$ => $p \cdot q_1 < p \cdot q_2$ => $t (p \cdot q_1) \leq t (p \cdot q_2)$ => $t (q_1 /p) \leq t (q_2 /p)$; 2) If q is an 'a priori' anti-prototype, that is t (q)=0, it can be supposed that it is also an 'a posteriori' working-prototype, since p · q < q => t (p · q) ≤ t(q) = 0, that implies t(p · q) = 0, and it could be defined t (q/p) = 0. Nevertheless, a different kind of problem arises when q is an 'a priori' prototype, since from t (q) =1 it is not immediate to arrive at, or to define, t (q/p) = 1, and some sufficient conditions seem necessary for it. One of such conditions is the validity for the pair (q, p) of the semi-additive law t (q · p) + t (q + p) = t (q) + t (p), since then: q < q + p => t (q + p) = 1, it follows t (q · p) = t (p), and finally t (q/p) = t (p)/t (p) = 1.

It should be remarked that 'a priori' values are not always attainable from past historical numerical records; this is the case in some linguistic cases, like the before mentioned 'Candidate X will win the election' when, for instance, such a candidate participates in an election for the first time. In such cases, 'a priori' values could be approximate through the experiences, or subjective evidences, the observer can (directly or indirectly) have on the corresponding situation; newly, the contextual knowledge can conduct to attribute a value to the 'a priori' ones, and only if there is a contextual ignorance it can be accepted to take the value 0.5, since t (p) = t(p') implies, provided t(p') = 1 – t(p), t (p) = 1/2. If the previous evidence is represented by e, the former formula still can be written in the form t (q/p ·e) =: t_e (p < q) = t (q · (p · e))/t (p · e), and only in the ignorance's case it is acceptable to take t (p · e) = 0.5 that, since it is q · (p · e) < p · e, implies t (q · (p · e))) ≤ t (p · e) = 0,5, and t_e (p < q) ≤ 1, that informs on nothing.

Actually, and for instance, using additional information to specify either a measure m for a predicate P, or a degree t for the validity of a linguistic expression, is nothing else than 'a priori' information perhaps supplied by an expert, and not always coming from numeric historical records; hence it is, in a Bayes-style line, that is necessary for the specification of measures. The use of additional 'a priori' information seems to be a proper and unavoidable resource for the mathematical modeling in several aspects of commonsense reasoning.

Nevertheless, the measuring of the extent up to which a conditional can be true is indeed an open problem when no numerical and historical records are known. It should be observed that supposed it holds $p \cdot q < q$, and hence $t (p < q) \leq t (q)$, from the former expression follows the upper bound $t (p < q) \leq \min (1, t (q) / t (p))$ for the measure of the conditional. Were p irregular, $t (p) = 1$, implies $t (p < q) \leq t (q)$, and $t (p) = 0$, implies $t (p < q) \leq 1$, actually allowing $t (p < q)$ to be either 0 or 1. Notice that taking $t (p < q) = \min (1, t (p) / t (q))$, it follows

$$t (p < q) = t (q), \text{ if } t (p) = 1,$$

and

$$t (p < q) = 1, \text{ if } t (p) = 0,$$

a result coincidental with that obtained by interpreting $p < q$ by $p' + q$, and $t (p < q) = \max (1- t(p), t(q))$, as it is done in the classical case.

The representation and the measuring of conditionals is, in its general aspects, an open problem whose study only can come from experimenting in the context-dependent and purpose-driven praxis, in natural language and ordinary reasoning.

3. An almost open subject of some relevance is still the following. Suppose that (contextual) degrees of true of the premises, $t (p_1), \ldots, t (p_n)$, are known, and that q is a conjecture of such premises. How a (contextual) numerical index of the support q inherits from the premises could be defined? Which is the numerical support, or weight w (q), conjectures q inherit from premises [34]? The subject is relevant since p </q' does not allow to immediately link t (p) and t (q).

3-a. A formula for w (q), in some way inspired in both the old idea of the *logos* as *ratio,* and in the modern technical idea of *efficiency* of electrical power generators, can be defined by the quotient:

$$w (q) = [t (p^+ \cdot q) - t (p^+ \cdot q')] / [t (p^+ \cdot q) + t (p^+ \cdot q')] \in [-1, +1],$$

where $p^+ = p_1 + (p_2 + (p_3 + (\ldots) + p_n)\ldots)$, is the disjunction of the premises once taken in the same order previously adopted for the résumé p, and with which, supposed < is transitive and since then $p < p_i$, and $p_i < p^+$, imply $p < p^+$. The term with the minus sign tries to show the amount of support strictly in favor of q, and that with the sign plus tries to measure all the support that can be known. Thus,

1) Provided it is $(q')' \approx q$ (strong negation), then $w(q) = - w(q')$.

2) $w(q) = 0 \Leftrightarrow t(p^+ \cdot q) = t(p^+ \cdot q')$, the premises carry the same degree of truth pro and against the conjecture.

3) $w(q) = -1 \Leftrightarrow t(p^+ \cdot q) = 0$, the premises carry no truth pro the conjecture.

4) $w(q) = +1 \Leftrightarrow t(p^+ \cdot q') = 0$, the premises carry no truth against the conjecture.

5) $-1 \leq w(q) \leq 0 \Leftrightarrow t(p^+ \cdot q) \leq t(p^+ \cdot q')$, the premises carry more truth against than pro the conjecture.

6) $0 \leq w(q) \leq +1 \Leftrightarrow t(p^+ \cdot q') \leq t(p^+ \cdot q)$, the premises carry more truth pro than against the conjecture.

7) It should be remarked that in general $q_1 < q_2$ does not imply $w(q_1) \leq w(q_2)$: usually w cannot be understood as a measure of true for ql; it simply reflects a numerical weight of the support q inherits from the premises.

3-b. It should be noticed that if all the premises are irregular, and t is a probability, since in such a Boolean situation it is $p^+ = p^+ \cdot q + p^+ \cdot q'$, follows $t(p^*) = t(p^+ \cdot q) + t(p^+ \cdot q')$, and then: $w(q) = [2t(p^+ \cdot q) / t(p^+)] - 1 = 2t(q/p^+) - 1$, w (q) is just the conditional probability after the scale changing from [0, 1] to [-1, +1] given by twice minus one. Consequently, the conditional probability can be seen as $(1 + w(q))/2$, just coming from the probabilistic support inherited by q from the premises.

The following example in the Boolean algebra of the subsets in the interval [0, 10], with p_1 = 'numbers between 2 and 6', p_2 = 'numbers between 3 and 5', and p_3 = 'numbers between 4 and 7', can be in order to show that the weight w does not well enough discriminate between the several types of the reasoning conclusions. With these statements, once specified by the sets [2, 6], [3, 5], and [4, 7] respectively, it is $p^+ = [2, 7]$, and p = [4 5], and taking $t([a, b]) = |b - a| / |a + b|$, it follows:

- If q = ‘numbers between 2 and 5’ ([2, 5]), since it is q ⊆ p, q is a hypothesis, and w (q) = + 0.2.
- If q = ‘numbers between 3 and 6’ ([3, 6]), since it is p ⊆ q, q is a consequence, and w (q) = + 0.2.
- If q = ‘numbers strictly greater than 2 and less than 4’ ((2, 4]), since p ⊆ q’, q is a refutation and w (q) = - 0.2.
- Finally, with q = ‘numbers between 3 and 8’ ([3, 8]), q is just a speculation and w (q) = + 0.6.

Notwithstanding all that, the problem of defining a numerical index translating up to which extent the truth of the premises supports the conjectures or refutations suitability is not, actually, a solved question.

4. It even still lacks a final, and also not conclusive, comment on t (p < q), and t (q < p), that is, in the question of the truth of the inverted conditional. The problem is, in general, an open one and it is only solved in some particular cases.

4-a. Provided p and q are irregular statements, and t is a probability. In this case, and expressing t (p<q) by the Bayes formula, it follows t (p<q) = t (q<p). [t ((p) / t (q)], as it have been proved. Hence,

- t (p<q) ≤ t (q<p) ⇔ t (p) ≥ t (q), the measure of the inverted conditional is bigger than that of the original, if and only if that of its antecedent is smaller than that of its consequent.

- t (p < q) ≥ t (q<p) ⇔ t (p) ≤ t (q), the measure of the inverted conditional is smaller than that of the original, if and only if that of its consequent is smaller than that of its antecedent.

Thus, it is: t (p<q) = t (q<p) ⇔ t (p) = t (q), the measure of the inverted conditional equals that of the original if and only if its antecedent and consequent do have the same measure.

The same result is reached, even if p and q are regular statements, if it were taken t_1 (p<q) = min (1, t (p)/t (q)), but not if t_2 (p<q) = max (1- t (p), t (q)), since, for instance with t (p) = 0.3 ≤ 0.7 = t (q), is t_2 (p<q) = 0.7 ≥ 0.3 = t_2 (q<p). In the first case, it is t_1 (p<q) = 1 ⇔ t (q) ≤ t (p), and in the second it is t_2 (p<q) = 1 ⇔ t (p) = 0, or t (q) = 1.

4-b. It seems that the truth of the inverted conditional is linked with the meaning that can be attributed to its original conditional; the relationship existing between p<q and q<p is not a universal one, but it is context dependent.

5. As it has been formerly said, if q is a consequence of p, $p < q$, then from $t(p) \leq t(q)$ follows that its degree of true should be, at least, that of p. If q is a hypothesis of p, $q < p$, from $t(q) \leq t(p)$, follows that its degree of true cannot be bigger than that of p. Provided the structure on which p and q are inscribed in is a Boolean algebra, since $p \leq q \Leftrightarrow p \cdot q = p$, it follows prob (q/p) = 1; and if $q \leq p \Leftrightarrow p \cdot q = q$, it follows prob (q/p) = prob (q) / prob (p). If q is a speculation such that $q' \leq p$, from $p = p \cdot q + p \cdot q' = p \cdot q + q' \Rightarrow$ prob (p) = prob ($p \cdot q$) + 1 − prob (q), follows prob (q/p) = 1 + [(prob (q) − 1)] / prob (p). In the case of a creative, or wild speculation, in which q is not comparable neither with p, nor with p', there is no way of finding such a formula and it remains an open question

V Conclusions

1. There is a general agreement among philosophers and scientists that rationality is a big, if not the main, difference between *Homo* and the other species on Earth. It seems very difficult to believe that rationality is actually possible without representing and processing (thanks to the flexibility of natural languages and ordinary reasoning) what arrives to the brain through the senses; both language and reasoning seem to be two characteristics making of *Homo* a species able to foresee, to progress and to spread all over the Earth. Without its capability for making analogies and conjectures, for guessing, today *Homo* will remain, probably and in the best case, closed in some more or less huge regions where predators could not extinguish the species, and like it happened with many other of them. In general it can be said that rationality lies in the human linguistic capability of doing conjectures and refutations that, often enough, are done thanks to analogy.

Even more, it also seems that the learning of language boosters reasoning, and it is also difficult to know if once language is acquired jointly with the world's personal experience and reasoning, it remains truly 'natural' or it is actually influenced with the many symbolic representations to which people is exposed from the very early; hence it can be, perhaps, less natural than it seems. All this, nevertheless, only can arrive to be well known through the experimental methods of natural sciences; possibly, with only 'pure thinking' alone never will be clarified all what is involved with natural language and ordinary, non-formally deductive, reasoning. Anyway, speculating on these subjects is relevant in itself for, at least, finding a way towards such clarification; without speculation there is never room enough for knowledge.

2. Both meaning[34] and deductive reasoning [36] deserved a lot of serious studies in Semantics, Mathematics, and Philosophy; in particular, logical consequences were well defined by Alfred Tarski [35], and a good idea on the

meaning of words was suggested by Ludwig Wittgenstein [31], and related with the use of words and with family resemblances. Even if not mathematically so well defined as consequences were, also hypotheses did receive great attention in Philosophy [30] and Science [32]. Nevertheless, what was not considered up to now are the representing and measuring of meaning, and the concept of speculation, both seeming to be necessary for starting with a new scientific way towards introducing experimental methods in the fields of natural language and ordinary reasoning, by also looking at Computer Science and, in particular, at Artificial Intelligence. A way that, at its turn and based in symbolic representations, can be able to open a window to new philosophical reflections on these mixed subjects without charging them with too many and doubtless laws, like those usually presumed in the classical calculus of logic, but, additionally, allowing to see which laws should be presumed for arriving at some results that could be formerly expected. This paper turns around ideas on representing qualitative meaning in a naïve and simple symbolic frame, on specifying measures for it, as well as on the classification of the different kinds of ordinary or common sense reasoning as just conjectures and refutations. Within this framework some concepts, like that self-contradictions only can be inferred from self-contradictions, that the refutation of hypotheses, or its reaching by conjunction between the premises' résumé and a speculation, can be formalized and proven. It is also shown how is that the reasoning by analogy can be seen as a natural and powerful instrument for arriving at conjectures and refutations, and that there are actually classical logic's situations in which the conclusions of a inductive reasoning can be heuristically reachable by a combination of forwards and backwards deductions once some information on the résumé and the conclusion is known.

All that requires, of course and at each case, to interpret the inference relation $<$ in an interpretable form, as it is, in Boolean algebras to take $<$ as its lattice's order $\leq$ ($p \leq q \Leftrightarrow p \cdot q = p$), since $p' + q = 1$ is equivalent to $p \leq q$. That is, asserting 'If p, then q' can be seen, in a Boolean algebra and through its equivalence with 'not p or q', as $p \leq q$. It requires to represent 'asserting q' by 'q is a tautology', that in Boolean algebras just means $q = 1$.

- [It should be noticed that, actually, this paper supplies the study of natural language and ordinary reasoning, with two concepts not considered before. The first is that, since the triplet $(X, <_p, m_p)$ is but a quantity, the full meaning, or sense, of a predicate P in a universe of discourse X can be viewed as, or represented by, a quantity. The second is the introduction of the, in principle non-deductive conjectures, called speculations, and their classification in two types of which the second, being properly non-monotonic, is where the branch of Artificial Intelligence called 'Non-monotonic Reasoning' can be inscribed in. What could follow from these new concepts is still and in the main, a true open question].

3. A relevant subject concerns the linguistic use of the word ‘probable’, whose measuring requires to know its qualitative meaning before designing the corresponding measures. If, for the (classical or irregular) predicate ‘probable’ it seems without doubt that the qualitative relation $<_{probable}$ (less probable than) can be identified with the inclusion of sets, for regular predicates the cardinality of whose measures is of a doubtful character, this is not so obvious and further study on it is waiting to be done. In any case, it is also dubious that the measures of the linguistic word ‘probable’ should be always ‘probabilities’ since its additive law does not seem to be universally applicable, in part, since the ‘growing’ of the relation $<_{probable}$ rarely can be seen in the form of a classical set’s partition. It requires the selection, or design, of particular types of imprecise conjunctions and disjunctions for which there actually can be understood what an ‘empty disjunction’ is. For instance, to measure with a probability ‘x is small is probable’ in [0, 10], requires to define under which disjunction (+), and conjunction (·) it can be said that ‘x is small or y is small’ is such that ‘x is small and y is small’ is either empty, or at least contradictory; that is when it is either $(\mu_{small} \cdot \mu_{small})\,(x) = 0$, or $\mu_{small}\,(x) \leq \mu_{small'}\,(x)$, for all x in [0, 10]. This is an open problem whose solution could, eventually, shed some light on the debate between those that hold either a ‘frequentist’, or a ‘subjectivist’, point of view on probability.

4. A last concept on which a reflection is in order is that of Truth. It is an abstract concept coming from the practical application of the predicate ‘true’ to the universes of linguistic discourse, and its opposite ‘false’, not always coincidental with the use of the negation ‘not true’. To capture the qualitative meaning of ‘true’ it is necessary the knowledge of the empirical relation $<_{true}$ (less true than) in the corresponding universe X, and to capture a full meaning a triplet $(X, <_{true}, m_{true})$, with which the degree up to which x in X is true is $m_{true}\,(x)$ ϵ [0, 1]. This shows that ‘true’ is context-dependent and often purpose-driven; hence, there is in principle no universal way of applying such predicate in the language. Even in the case of an irregular use of ‘true’, its single use is limited to the corresponding universe; for instance, in a statement like ‘five is a prime number is true’, it just means that five has no more divisors than 1 and 5, and the use of true cannot be imported to statements like ‘5 +5i is a complex number’ since complex numbers do have a different definition than positive integer and primes. In a process of reasoning, the idea of measuring statements like ‘x is P is true’ should combine the measuring of P with that of ‘true’, in the form $t\,(x \text{ is P is true}) = m_{true}\,(m_P(x))$, that is by means of the composite function $m_{true} \circ m_P$ that just will coincide with m_P provided m_{true} is the identity, the only case in which it is justified to say that $m_P\,(x)$ is the degree of true of ‘x is P’. The predicate ‘true’ is so charged with a terrible history for what concerns believing, or faith, that it should be taken with great care; it is never to be forgotten than in language and commonsense reasoning ‘true’ has never a single regular use.

5. This paper only tries to be a first attempt towards a mixing of a philosophical reflection with symbolic representation around the relevant subjects of natural language and commonsense reasoning, and that comes from the wish of the authors of not separating the philosophical reflection from that typical of science. Of not falling in just metaphysical reflections that, if they don't completely refuse by appreciating its suggestive power, consider insufficient for the purpose of analyzing those subjects; today and, for instance, to undo the 'Gordian Knot' of Artificial Intelligence, a new and experimental way of studying natural language and commonsense reasoning with the help of measuring seems to be essential. Natural language and commonsense reasoning are but natural phenomena that, like all the other natural phenomena, deserve of a scientific study that, notwithstanding and since all humans actually have is natural language and ordinary reasoning, cannot be totally unlinked with 'reflection', with philosophy.

For such a goal a first glance at those subjects with the naïve idea of 'Let's look what is practically done here', and by keeping ourselves as far as possible from all the theoretical constraints usually imposed by contemporary logic, is basic for not imposing laws whose validity is more than doubtful in language and reasoning. Notice the scarce number of properties that have been supposed for (·), (+), ('), etc., and that even transitivity is not universally supposed, but just accepted for the corresponding n-tuples. At the end, logic is mainly devoted to study formal deductive reasoning with the help of artificial languages, and, also at the end and presumably, it was an empirical way what was taken into account in the new point of view, Ludwig Wittgenstein tried to adopt in his strong passing from the pure thinking in the 'Tractatus Logico Philosophicus', to his empirical thinking in the 'Philosophical Investigations' that, possibly motivated by his own experiences in elementary teaching and his early formation as an engineer, today can be seen as an antecedent towards a new experimental way for studying language and reasoning.

Let us end with a question on whose answer could lie the possible relevance of what is considered in this paper: Can a human being unable to represent, play with words or speak, and conjecture, be imagined?

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