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Fuzziness: towards a scientific domestication of vagueness?*

Enric Trillas[1]

[1] Oviedo, Spain

Email: etrillasetrillas@gmail.com

*To Professor Settimo Termini, remembering our forty years of friendship.

1 Introduction

Historically, there have been some problems that initially were philosophically posed, either in mystical or in metaphysical terms but that, latter on and step-by-step, were translated into scientific terms and through a process of successive clarifications allowing to arrive, finally, at a scientific theory in which its more essential treats are clearly defined and facilitates its measuring. That is, by moving the subject into Science, where it is not enough the reflexive and critical thinking that is typical of philosophy, but is necessary the reasoning in the finest form than possible, and under the hard exigencies of the scientific method; are processes conducting to the scientific domestication of an old problem. Notwithstanding, it should be openly recognized that if the initial philosophical worry for a problem facilitates its further reconsideration at the light of Science also, a posterior philosophically critical reflection is of a top value. Actually, the divorce between Science and Philosophy is not good; scientists and philosophers should be encouraged to strongly interact for, at least, benefitting the advance of culture and its bring up to date.

Usually, former advances in both the practical methods for observing reality, and the theoretical terms able to translate the empirical knowledge into a scientific theory were required for that domestication, for passing from a jungle of vague and incomplete ideas to a garden of clear and measurable concepts. It happened so, for instance, with the problem concerning the Heavens Sun's System and along the long way going from the old first mystical ideas, to the scientific works of Ptolomy, Copernicus, Kepler and, finally, to Newton's theory; always in correspondence with the evolution of observation at nude-eye up to that by telescope, and each time facilitating best collections of observational data [1].

In an analogous form it can be seen the problem of reasoning with precise terms, from Aristotle to some Franciscan Scholastic philosophers like Ramon Llull, to Augustus de Morgan and George Boole [2] that, finally and thanks to the work of some Boole's mathematical 'clarifiers' like Ernst Schröder [3], allowed to arrive at Boolean algebras as the undisputable mathematical model for the reasoning with precise terms of which, at least potentially, all relevant information could be available. Concerning the reasoning with imprecise terms, of which 'Sorites' type arguments show they cannot be represented by sets (that Boolean algebras are not useful for dealing with them [8]), the algebras of fuzzy sets Lotfi A. Zadeh introduced in 1965 [4] proved they can be seen as a good enough mathematical model for it; also and for what concerns to lacks of information, the 1978 fuzzy sets

based Zadeh's theory of possibility [5] is a model to be considered.

Linguistic *vagueness* [6] is still, widely and mainly studied in philosophical as well as in logical terms, and it is today possible to make some preliminary advance towards a new scientific domestication of the linguistic phenomenon the word 'vagueness' names. A phenomenon that is pervasive in plain language and from which mathematical logicians largely tried to escape, but whose scientific domestication is necessary for dealing with ordinary reasoning, either deductive or inductive and expressed in plain language; the so-called people's commonsense reasoning, whose modeling is of a great relevance for Artificial Intelligence.

This paper just tries to shortly present how such domestication can be possible through the new concept of *fuzziness,* thanks to what Zadeh introduced and once showing that a fuzzy set is not, in itself, a mathematical function but a hazy entity that, nevertheless, is well anchored in plain language and appears in different states with imprecise words whose meaning is measurable in the universe of discourse, and for which each measure (or membership function) depends on the context in which such words are currently used, as well as on the persecuted purpose for such use. In short, words whose meaning is situational.

2 On the meaning of plain words

2.1

When a word P acting in a universe of discourse X (provided it is a set) can be recognized as well managed? Of course, when it is firstly known to which elements x in X is P applicable, and secondly when the variation of such application along X is captured. Hence, the meaning of P in X depends on how the elemental statements 'x is P' are used; in Wittgenstein own words, *the meaning of a word is its use in language* [7]. Provided it is known that P can be applied to x in X, that is, the statement 'x is P' has sense, then it still lacks to know its difference with whatsoever statement 'y is P' also with y in X; on the contrary, it cannot be said that the use of P in X is well captured.

Hence, it should be known that 'x is less P than y', or that 'y is less P than x', or that there is no relation between 'x is P' and 'y is P', that x and y are not comparable under P. It should be noticed that 'x is less P than y' just reflects that the property named by P, x and y exhibit, is less shown by x than by y; in addition, 'x is less P than y' and 'y is more P than x' can be confused. Usually, and except in those cases in which the property is given by an 'if and only if' definition, the relation 'less than' is empirically recognized.

Consequently, the binary relation $<_P \subseteq X \times X$ that, defined in X by [8]:

> x $<_P$ y if and only if 'x is less P than y', or, equivalently, 'y is more P than x', y $<_P^{-1}$ x,

can be seen as a mathematical way of translating into a graph (X, $<_P$) the more simple empirical knowledge available on the use of P in X. Such graph is called a *qualitative meaning* of P in X; indeed, it is not stated that the qualitative meaning is unique since each graph corresponds with an (observable) qualitative meaning. In the graph there can be, or not to be, minimal elements (those x for which no z ≠ x such that z $<_P$ x exists), and maximal elements (those x for which no z ≠ x such that x $<_P$ z exists); when there is just one minimal it is the graph's minimum, and when it is just one maximal is the graph's maximum. The only presumed property for the graph is the reflexive one, that is, x $<_P$ x for all x in X; hence, the minimum graph is that in which $<_P$ is the subset {(x, x); x in X} of X x X, and provided it is strictly contained in $<_P$, P is called *measurable* in X. Were it $<_P = \emptyset$, P is called *metaphysical* in X. Once a graph is fixed it is possible to go ahead.

2.2

F If P is measurable in X, a measure for its meaning is a mapping m_P: X →[0, 1], such that [8],

1) If x $<_P$ y, then m_P (x) ⩽ m_P (y), with ⩽ the linear order of the unit interval
2) If z is minimal, then m_P (z) = 0, and
3) If z is maximal, then m_P (z) = 1.

In general, and like it happens in other situations (for instance, with probabilities in a Boolean algebra), the former three axioms don't allow to specify a single measure, and, at each particular problem more information on m_P is necessary for specifying one. Usually, this information comes either from the context in which the corresponding problem is inscribed in, or from a set of reasonable hypotheses made on the shape of the mapping m_P; that is, each specification of m_P should be obtained through a process of design. Once a measure m_P is designed, the triplet or scalar magnitude (X, $<_P$, m_P), represents a *quantitative meaning* of P in X.

3 On what is a fuzzy set

3.1.

A couple of examples can be suitable to approach what is a Zadeh's fuzzy set. In the universe of discourse constituted by the set of censed inhabitants of London, let's consider the word 'young', and suppose we can recognize if a Londoner (John, Sally, Peter, etc.,) is or is not young, that is, considering and affirming statements like 'John is young', 'Sally is not young' or 'Sally is old', as well as relational statements like 'John is less young that Peter', etc. It is after this, and not before, that we consider and openly talk of the *Young Londoners,* that we grant entity to what *Young Londoners* denotes. Analogously, in the universe of discourse [0, 100], the word 'small' conducts to consider the entity *Small numbers* in such interval. Since neither 'Young Londoners', nor 'Small numbers', are subsets of the universe (as it can be shown by means of a Sorites type argument), what are these hazy entities? Accepting that its existence is warranted, at least in plain language, by its use in it, that they are well rooted or anchored in plain language, that are linguistic entities, it can be said they are *linguistic collectives*, or *fuzzy sets* (in Zadeh's terminology) [8]. A fuzzy set in X is but a linguistic entity, a virtual collective that arises, is generated, by the human mind from the use of a word P in X, the 'linguistic label' (in Zadeh's terminology) of the fuzzy set.

Hence, fuzzy sets or linguistic collectives are generated in a universe of discourse by a word whose qualitative meaning is previously known. Since nothing else seems to be necessary for considering 'the linguistic collective of the Ps in X', the 'fuzzy set labeled P' *can be identified* with (or at least *represented* by) a qualitative meaning, given by specifying a graph (X, $<_P$). In mathematical terms, a fuzzy set is but the graph making the word measurable and that, as it was formerly said, can be measured by several meaning's measures m_P, each one conducting to a (functional or numerical) 'state' translated into the scalar magnitude (X, $<_P$, m_P); in the Zadeh's terminology, m_P is called the 'membership function' to the fuzzy set labeled P in X [4], and it should be designed accordingly with the available contextual information related to the current problem, or by introducing some reasonable hypotheses on its shape. Hence, ***a*** membership function cannot be confused with ***the*** fuzzy set that, anyway, depends on the complete knowledge of $<_P$; the first just reflects a particular view of the second, and is facilitated by the context under which P is actually used in X. It should be noticed that a wrong design of m_P can conduct to represent a problem that is not the current one; membership functions must be carefully designed [9]; in addition, there can exist elements x at which, without being a maximal or a minimal, the measure m_P takes the values one or zero.

3.2.

The use of P in X is 'precise', or P is precise in X, when the relation $<_P$ reduces to the one just reflecting 'x is equally P than y', in symbols $x =_P y$. That is, since it is $=_P = <_P \cap <_P^{-1}$, when $=_P \; = <_P$, then the behavior of $<_P$ is constant in X [8]. Since,

$$x =_P y \Leftrightarrow x <_P y \;\&\; y <_P x \Rightarrow m_P(x) \leqslant m_P(y) \;\&\; m_P(y) \leqslant m_P(x) \Leftrightarrow$$
$$\Leftrightarrow \; m_P(x) = m_P(y),$$

all measure preserves 'P-equality' and, consequently, in the precise case it can only take two values; since $=_P$ classifies X in two disjoint subsets, one consisting in those elements linked by the symbol $=_P$, and the other consisting in those that are not (except with themselves), the measure will take only the values 1 and 0. It is the only measure, the Characteristic function of the subset {x; 'x is P'} in X. The imprecise use of P is revealed by the existence of at least an element x in which the measure neither takes the value one, nor the zero, but a value strictly intermediate between 0 and 1. It should be pointed out that, for all x in X, it is $x =_P x$ as a consequence of the reflexive law presumed for $<_P$.

Notice that since $x <_P y \Rightarrow m_P(x) \leqslant m_P(y)$, defining the new relation $<_{mP}$ by:

$$x <_{mP} y \Leftrightarrow m_P(x) \leqslant m_P(y),$$

it follows $<_P \subseteq <_{mP}$, that is, the 'act of measuring' can change the qualitative meaning, can enlarge it by adding more arcs to the graph, modify the fuzzy set; make it 'bigger'. Consequently, and as usually the practitioner of fuzzy sets just work directly with a membership function, he/she risks to confuse both relations $<_P$ and $<_{mP}$, but there can be elements in X not comparable under $<_P$, and since $<_{mP}$ is a part of the linear, or total, ordering of the unit interval, under it there are not incomparable elements. The risk comes from an imperfect knowledge of the fuzzy set; of only looking at it with just one of its contextual measures.

4 The big jump

4.1.

Many people working in fuzzy logic, identify the statements 'P is imprecise in X' and 'P is vague in X'. It should be recalled, nevertheless, that fuzzy logic always supposes P is measurable in X, that is, its behavior in X is always considered through a (previously designed) scalar magnitude $(X, <_P, m_P)$; it is presumed that P is not metaphysical in X, that it has, at least, a

membership function. The vagueness fuzzy logic can manage is that shown by measurable, meaningful, words. Hence, can be 'fuzziness' seen as 'measurable vagueness'?

As early as it was in 1972, Aldo de Luca and Settimo Termini did introduce 'fuzzy entropies' for measuring up to which extend a measurable word P can be seen as being *fuzzy* [10]. Even if they introduced the new concept in other words, it is a fact that de Luca and Termini begun from analyzing how 'P is less fuzzy than Q' can be defined in X through the respective measures m_P and m_Q; these authors defined, for instance, the measures of fuzziness by calling them 'fuzzy entropies' and mainly because there is one of them analogous to the well-known Claude E. Shannon's probabilistic entropy. To such goal, they did define the relation $<_{fuzzy}$ in an indirect form by newly ordering the unit interval through what they called its 'sharpened order' [10]:

$$r <_s s \Leftrightarrow 0 \leqslant r \leqslant s \leqslant 0.5, \text{ or } 0.5 \leqslant s \leqslant r \leqslant 1,$$

that is a partial but not a total order (for instance, 0.4 and 0.7 are not comparable), with just the maximum 0.5 and the two minimal elements 0 and 1; it is like breaking down the unit interval by the point 0.5, and changing the sense of the old linear order between 0.5 and 1. With such ordering they defined:

$$P <_{fuzzy} Q \Leftrightarrow m_P <_s m_Q \Leftrightarrow m_P(x) <_s m_P(x), \text{ for all } x \text{ in } X.$$

by trying to reflect that the values of m_P are closer to 0 and 1 (those of the precise or binary case) than those of m_Q. De Luca and Termini translated 'less fuzzy than' into 'close to precise', and it should be observed that if the breaking point 0.5 corresponds with the fix-point of the usual strong negation function 1 – x, it can be easily changed by any other point different from the extremes 0 and 1, since there are always a lot of strong negations with a previously established fix-point. De Luca and Termini's formulation don't depends, actually, on the particular type of the involved linguistic negation provided it is strong, and it is open to study the case in which the negation is not strong [8]. It is also opened the case in which more general measures of fuzziness are taken into account, as for instance those de Luca and Termini further presented in their papers [13, 14].

4.2.

Once the respective membership functions are made comparable, that is, once in a universe of measurable words that can be considered in X, let´s

write it W (X), is known the fuzzy set (W (X), $<_{fuzzy}$), the linguistic collective of 'fuzzy words', it just lacks [11] to define its states, that is, adding the corresponding measures for the meaning of 'fuzzy' as a predicate for the words in W (X). That is, mappings m_{fuzzy}: W (X) → [0, 1], such that:

1) It exists a function E ('fuzzy entropy') such that $m_{fuzzy}(P) = E \circ m_P$,
2) If $P <_{fuzzy} Q \Leftrightarrow m_P <_s m_Q \Rightarrow m_{fuzzy}(P) <_{fuzzy} m_{fuzzy}(Q) \Leftrightarrow E \circ m_P <_s E \circ m_Q$,
3) If $m_P (x) = 0.5$ for all x In X, then $m_{fuzzy} (P) = 1$.
4) If $m_P (x) \in \{0, 1\}$, for all x in X, then $m_{fuzzy} (P) = 0$.

Notice that those functions E are mappings $[0, 1]^X \rightarrow [0, 1]^X$ that should be designed at each case, and depending on the wished characteristics for the corresponding problem. Of course, this functions can be expressed by mappings E: [0, 1] → [0, 1] and defining $(E \circ m_P) (x) = E (m_P (x))$, for all x in X. Notice also that m_{fuzzy} don't necessarily needs to be defined through a fuzzy entropy E; it can be directly considered (and, of course, with more difficulty) without the functional expressivity's hypothesis shown by axiom (1). Notice also that the former axioms strongly depend on the design of the measures corresponding to the involved words.

In this way, the word 'fuzzy' itself is made measurable in W (X), generates the fuzzy set of fuzzy words, the new concept of *fuzziness* as the restriction of *vagueness* to measurable words, arises and opens the door to a scientific domestication of vagueness, as well as to shortly state that Zadeh's fuzzy set theory consists in trying to scientifically studying fuzziness.

5 Conclusions

In general, it seems very difficult to establish when a word P is 'less vague' than another Q. For instance, with the (Hardy's) vague term 'round' applied to natural numbers that generates the linguistic collective/fuzzy set of 'round numbers', it is not at all clear when a natural number is less round than another. It is for this reason that the 'roundness' of natural numbers was initially studied without previously clarifying which is the relation $<_{round}$ but by directly going to design a membership function associating to each natural number a degree of roundness, just attending to the exponents of the prime numbers appearing in its prime factors expression [12]. That is, escaping from previously establishing a graph (N, $<_{round}$), from explicitly fixing a qualitative meaning of 'round', and just considering a pair (N, m_{round}); something that, as it was said, comports the design's typical risk of committing a 'mistake' by directly identifying $<_{round}$ with $<_{mround}$. That is, by

perhaps enlarging the 'intuitive' meaning of round in the set N of natural numbers; by analyzing a linguistic collective 'bigger' than that the use of the word round (a small number of prime factors but with big exponents) actually generates.

De Luca and Termini did not commit such a 'mistake'. They start from previously representing the relation $<_{fuzzy}$ (certainly not among words, but among membership functions), and left open the mathematical problem of obtaining the more suitable fuzzy entropy E at each case; and as it can be seen in, for instance [16], there can be a lot of functions E. Their work represents the first time in which the meaning of a word is qualitatively analyzed before trying to measure it. Of course, as early as in 1972, and by not taking into account metaphysical concepts but measurable ones, the paper of de Luca and Termini was a solid root allowing to support a nice tree with both the branches of qualitative and quantitative meaning of words, as well as that allowing to consider the concept of fuzziness as a scientifically analyzable [15] restriction of the old concept of vagueness. Additionally, as an usual consequence, the measures of fuzziness were employed to select, for solving concrete problems, those membership functions showing, in a given context, the less than possible degree of fuzziness for trying to reduce imprecision; that is, those being the 'more precise' than possible in the corresponding problem. As always, the scientific domestication allowed applications that without the possibility of computing are not possible; in addition, such work allowed to newly viewing fuzzy sets as they are presented in this paper. In this way, all of that is more open to criticisms permitting to booster rationality up to some new step ahead.

When possible, it is very fruitful translating into scientific theories what earlier were mystic or metaphysical ideas. During the time's periods in which it is not possible, the right to doubt on their interest for, at least, the material benefit of people is open.

References

1. J-P. Luminet, 2007, El enigma de Copérnico. Ediciones B.
2. G. Boole, 1854, The Laws of Thought. Open Court & Macmillan.
3. E. Schröder,1905, Vorlesungen über die Algebra der Logik (Exakte Logik). Teubner.
4. L.A. Zadeh, 1965, 'Fuzzy Sets'. Information and Control, 8: 338-353.
5. L.A. Zadeh, 1978, 'Fuzzy Sets as a Basis for a Theory of Possibility'. Fuzzy Sets and Systems, 1: 3-28.

6. K. Fine, 1966, Vagueness, Truth, and Logic. Synthese, 54: 119-150-

7. L. Wittgenstein, 1958, Philosophical Investigations. Basil Blackwell.

8. E. Trillas, 2017, On the Logos: A Naïve View of Fuzzy Logic and Ordinary Reasoning. Springer.

9. E. Trillas, S. Guadarrama, 2010, 'Fuzzy representations need a careful design'. International Journal of General Systems, 39/3: 320-346.

10. A. de Luca, S. Termini, 1972, 'A definition of a non-probabilistic entropy in the setting of fuzzy sets theory'. Information and Control, 20: 301-312.

11. E. Trillas, 2015, 'A scrutiny on globally measuring imprecision'. Alla Ricerca! (Liber Amicorum to honour Settimo Termini), Eds, P.Greco et altri: 193-200. Doppiavoce.

12. A. Sobrino, E. Renedo, 2007, 'Round Numbers, revisited. A new fuzzy approach'. Fuzzy Sets and Systems, 158/14: 1618-1629.

13. A. de Luca, S. Termini, 1979, 'Entropy and Energy Measures in Fuzzy Set Theory'. Advances in Fuzzy Set Theory and Applications (Eds., M.M. Gupta, R.K. Ragade, R.R. Yager): 321-338.

14. A. de Luca, S. Termini, 1988, 'Entropy Measures in Fuzzy Set Theory'. Systems and Control Encyclopedia (Ed., M.G. Singh): 1467-1473.

15. E. Trillas, 2014, 'How Science domesticates concepts'. Archives for the Philosophy and History of Soft Computing, 2014/1: 1-23.

16. E. Trillas, T. Riera, 1978, 'Entropies in Finite Fuzzy Sets'. Information Sciences, 15: 158-168.