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A Brief History of Fuzzy Logic in the Czech Republic and the Significance of P. Hájek for Its Development

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Abstract In this paper, we will briefly look at the history of mathematical fuzzy logic in Czechoslovakia (and the Czech Republic) starting from the 1970s and extending until 2009. The role of P. Hájek in the development of fuzzy logic is especially emphasized.

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1 Introduction

This paper presents a brief look at the history of mathematical fuzzy logic in Czechoslovakia (and the Czech Republic) from the 1970s to 2009. We will especially emphasize the crucial role of P. Hájek in its development. Let us recall that in 1993, Czechoslovakia split into two countries: the Czech Republic and the Slovak Republic. Therefore, we will refer mostly to mathematical fuzzy logic in the Czech Republic.

This paper is founded on personal remembrances, and claims neither completeness nor objectivity. The list of publications presented accompanies our description and is far from being complete.

We focus on mathematical fuzzy logic and some closely related topics. Therefore, various other branches intensively studied in fuzzy set theory as well as in logic are not included.

It should be noted that some areas that we suppose to be less known are discussed in more detail. Finally, let us remark that occasionally we accompany the description of our memories with mathematical concepts or formulas to provide a more clear or specific explanation but that, because of different focus of this paper, we cannot be too detailed or exact.

2 The Seventies

There was no significant movement in life in Czechoslovakia in the seventies. Those years were negatively influenced by communist normalization after the year 1968, and so those who stayed in Czechoslovakia[1] were subject to various kinds of restrictions motivated by the communist party's attempt to have a voice in everything. This pressure was especially serious in the humanities. Luckily, in mathematics, the pressure was not as strong because orthodox enthusiastic communists were not scientists and so they could not understand abstract sciences such as mathematics. On the other hand, access to western literature was quite limited, mainly for economical reasons rather than political ones; Czechoslovakia simply had very limited sources of western currency, and thus, research was partly closed off from western science. Luckily, a lot of good western books were available in Russian because the Soviet Union at that time did not respect copyright laws and freely published most good English books in Russian.

With regard to fuzzy logic, two great ideas were raised in these gray days. The first idea was the concept of *Alternative Set Theory* (AST), and the second idea was establishing a precise formal system of fuzzy logic that is now known as *fuzzy logic with evaluated syntax*.

The basic concept in AST, with a motivation similar to that of the fuzzy set, is that of *semiset*. This is a proper class (not a set) X as a subclass of some set a, i.e.

$$X \subseteq a. \tag{1}$$

[1] Many thousands of mostly well educated people emigrated from Czechoslovakia in 1968-69.

As has been shown by P. Vopěnka and P. Hájek in the book [72], semisets can be consistently introduced inside classical Gödel-Bernays axiomatic set theory by slightly weakening its axioms. Note that semisets are by no means as strange as they might seem at first glance. For example, let us consider a class *Sm* of *small natural numbers*. Such numbers are 0, 1, ... but this sequence is not finished — there is no last small number and, therefore, also no first large number. On the other hand, there is surely a number n that is large (i.e., surely not small); for example, $n = 10^{10}$. Thus, we have $Sm \subseteq n$. Let us stress that *Sm* cannot be a set because then it would have to be finite (because n is finite) and so the last small number would have to exist. The class *Sm* is a typical semiset.

These considerations led P. Vopěnka to develop the AST theory. In about 1974, a group of logicians[2] had gathered around him and commenced developing a new set theory that it was thought might become an alternative (hence the name) to the classical Cantorian one (ST). Several important principles of AST depart from ST. The most important among them is a new model of infinity. In classical set theory, an infinite set is obtained via its never-ending extension by new elements, while in AST, the infinity emerges when one is dealing with sufficiently large but otherwise classically finite sets. We arrive at the concept of the so-called *natural infinity*, which, unlike ST, is directed inside: a set is infinite if it contains a proper subclass. In other words, a part of an infinite set is not a set. One can verify that a in (1), as well as the number 10^{10} in our example, is infinite. The crucial concept in AST is that of a semiset $\mathbb{FN}$ of all *finite natural numbers* that is a proper subclass of a class $\mathbb{N}$ of all natural numbers. This means that any number $\alpha \in \mathbb{N} - \mathbb{FN}$ is infinite. But recall again that such a number α is *classically finite* (though very large in the given context). We encountered this infinity phenomenon emerging inside very large sets. The semiset $\mathbb{FN}$ is *infinitely countable*, while $\mathbb{N}$ is *uncountable*.

In addition to the many papers published, mostly in the journal CMUC (Commentationes Mathematicae Universitatis Carolinae), the small book [73], written by P. Vopěnka, was published. It presents the essential results for AST up to 1979. Ten years later, a much more extensive book [74] presenting a wide array of well established philosophical background material for AST appeared. Unfortunately, it has never been translated into English. The most recent book on AST focusing especially on its philosophy is [75].

The reader has surely noticed that elements of the above-introduced semiset *Sm* represent numbers of grains that do not form a heap in the known *sorites paradox*:

> *One grain does not form a heap. Adding one grain to what is not yet a heap does not make a heap. Consequently, there are no heaps.*

This paradox is resolved in AST. Moreover, semisets represent vaguely defined classes: i.e., just the objects that are in fuzzy logic modeled by *fuzzy sets*.

The *fuzzy logic with evaluated syntax* $\mathrm{Ev}_{Ł}$ was developed by J. Pavelka in his PhD thesis defended in 1976 and published as a sequence of three papers in [63]. Let us

[2] Among other active people in this group, let us mention that K. Čuda, A. Sochor and J. Mlček.

emphasize that the main reviewer of the thesis was P. Hájek. This was probably his first encounter with mathematical fuzzy logic.

Pavelka's papers contain full descriptions of the formal system of propositional $\mathrm{Ev}_Ł$ including the algebraic analysis of possible structures of its truth values and its metatheory. The third of the papers [63] contains a complicated algebraic proof of the syntactico-semantical completeness of $\mathrm{Ev}_Ł$.

The second of the papers [63] contains a deep algebraic analysis of the possible structures of truth values and refers especially to the concept of residuated lattice, thus anticipating the theory of fuzzy logic developed later by P. Hájek. The basic algebra considered there is (what is now called) *standard Łukasiewicz MV-algebra*

$$\langle [0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1 \rangle \tag{2}$$

where $\otimes$ is the Łukasiewicz conjunction (also called the Łukasiewicz or bold product) defined by $a \otimes b = \max(0, a+b-1)$ and $\rightarrow$ is the Łukasiewicz implication defined by $a \rightarrow b = \min(1, 1-a+b)$, $a, b \in [0,1]$. Another structure considered is the finite Łukasiewicz chain (Pavalka proved that the infinitely countable chain is not acceptable).

The logic developed by J. Pavelka is consistent with the principal idea of fuzziness, namely not only by assuming truth values between 0 and 1 but also allowing axioms not to be fully convincing. This means that a formula A can be an axiom only in some (arbitrary) degree[3]. This assumption implies the concept of proof accompanied by a value that expresses the degree to which the proof is convincing. As a result, we arrive at the concept of *provability degree* (of a formula A). The prevailing principle in $\mathrm{Ev}_Ł$ is the *maximality principle* stating that if there are more proofs of a formula A with various degrees then the final provability degree is equal to the maximum (supremum) of all of the former ones. This also implies that all these degrees are equally important; essentially, if we cannot reach 1, then any other degree is satisfactory, provided that we know that it cannot be increased.

A very important consequence of metamathematical considerations about $\mathrm{Ev}_Ł$ is the theorem stating that, if we consider a residuated lattice on $[0,1]$ and the implication (residuation) operation is not continuous, then it is *impossible to form a syntactico-semantically complete fuzzy logic with evaluated syntax*. This theorem was originally proved by Pavelka in [63]; see also [53], Theorem 4.2. Consequently, the only plausible structure of truth values on $[0,1]$ for this logic is the standard Łukasiewicz MV-algebra and its isomorphs.

3 The Eighties

These years were characterized by increasing interest in the fuzzy set theory and fuzzy logic. A notable paper was that of A. Pultr [69] emphasizing the role of fuzzy equality

[3] In fact, all formulas are then axioms, but most of them only in the degree 0.

in fuzzy set theory where the essential transitivity axiom is stated as follows:

$$(a \sim b) \otimes (b \sim c) \leq a \sim c, \qquad a,b,c \in L.$$

By $x \sim y$, we denote the degree of equality between x and y, and $\otimes$ is the multiplication operation in a considered residuated lattice of truth values. This concept has been extensively developed in many papers and books in fuzzy set theory since then. Recall, among other ideas, the important concept of the *real semiset* in AST; i.e. a semisets that is fully characterized by some indiscernibility equivalence relation $\equiv$. The latter is obtained as the intersection of a decreasing countable sequence of set-definable relations[4]

$$\equiv = \bigcap \{R_n \mid n \in \mathbb{FN}\}. \tag{3}$$

This definition is motivated as follows: two objects are indiscernible if all the requirements for still higher and higher precision fail to be met. Hence, fuzzy sets can be seen as reasonable approximations of semisets when one is replacing the indiscernibility relation with the corresponding fuzzy equality. This idea is formally developed in [42].

It seems that the main impulse for the growing interest in fuzzy logic in Czechoslovakia came from the book [43] written by the author of this paper. This book was quite successful and thus was published as a revised English edition in [44] and then again in Czech [46]. It is important to emphasize that the fuzzy set theory is explained in this book in a unified way from the point of view of $\mathrm{Ev_Ł}$ (which is described in detail in Chapter 3 of the volume in question) as the metatheory at play. As a result, one can see that the concept of formal fuzzy mathematics as proposed much later (see below) is anticipated in this book.

At the same time, the author of this paper followed the ideas of J. Pavelka and developed a predicate version of $\mathrm{Ev_Ł}$ in his PhD thesis defended in 1988. Let us emphasize that the main reviewer of the thesis was again P. Hájek. The 1980s also represented years of intensive discussions between the author of this paper and P. Hájek about fuzzy logic and its meaning.

One part of the thesis was published in two papers [45]. It contains a proof (based on Pavelka's technique) of a generalization of the Gödel-Henkin completeness theorem:

$$T \vdash_a A \quad \text{iff} \quad T \models_a A$$

for all fuzzy theories T and formulas A where by $T \vdash_a A$ we mean that a formula A is *provable in a degree a* in T and by $T \models_a A$, that it is *true in a degree a*. Another crucial property proven there is a generalized deduction theorem:

$$T \cup \{1/A\} \vdash_a B \quad \text{iff} \quad \text{there is } n > 0 \text{ such that } T \vdash_a A^n \Rightarrow B. \tag{4}$$

Note that a slightly weaker form of this theorem is now known as the *local deduction theorem* in fuzzy logic (see below).

[4] Many technical details are omitted here. The interested reader should refer to [73].

It is worth mentioning that a working group for *Fuzzy Sets and Their Applications* was established within the Society of Czechoslovak Mathematicians and Physicists in 1985. This group has had annual meetings since then. The most important of its activities has been the *International Symposium on Fuzzy Approach to Reasoning and Decision-Making* (FARDM'90), which took place in a small town, Bechyně, in June 1990. The main organizers of this event were V. Novák, M. Mareš, J. Ramík, J. Nekola and M. Černý. The symposium had a key influence on the further development of fuzzy logic in Czechoslovakia. On November 17, 1989, the Czechoslovak communistic regime broke down. The country became open, people were allowed to travel freely and thus, closer contact with western scientists could be established. Because the freshly liberated communist country was interesting to visitors, many people took part in FARDM'90, coming both from the other communist countries and from western ones (the number of participants was slightly over 100). For the first time, we had an opportunity to discuss freely and to meet with the people whose papers we rarely had had the opportunity to read before. A large group of Russian scientists came (actually 27); furthermore, there were several people from Hungary, Poland, and East Germany, as well as D. Dubois from France, M. Roubens from Belgium, G. J. Klir, L. Kohout and J. Bezdek from the USA, among others . The impact of this symposium on fuzzy logic in the Czech Republic can still be felt today. Selected papers from this symposium have been published in the edited volume [48].

4 The Nineties

4.1 Beginning of the decade

This decade was probably the most active and important for the development of fuzzy logic not only in the Czech Republic but also on an international scale. After long and fruitless quarrels between Czech and Slovak representatives that delayed the economical and political progress of Czechoslovakia, it was decided that Czechoslovakia would be split up into two countries: the *Czech Republic* and *Slovakia* (the Slovak Republic). Though many people were regretful, this decision was very useful from the point of view of the development of both countries. At the same time, the existing close relations between scientists in the two nations were not harmed. Concerning fuzzy set theory, while Slovakia was more oriented towards algebraic and analytical aspects of fuzzy set theory, Czech Republic was more oriented towards fuzzy logic and logical foundations of fuzzy mathematics.

The PhD thesis by the author of this paper contained another result: namely a model of natural language semantics using fuzzy logic. The idea of developing such a model was suggested by P. Sgall (head of the Prague linguistic circle) and finally resulted in the book [47][5] that emerged as an unprecedented attempt at applying AST (hence the title) to the theory of linguistic semantics. The reason for choosing AST was the greater philosophical depth and the more sound and better justified tools in

[5]The book was published especially due to the kind support of G. J. Klir.

comparison with fuzzy set theory. While semisets seem to capture the core of the vagueness phenomenon, fuzzy sets in a certain sense seem to quantify it as if from the outside. The book contains a model of the meaning of the most important classes of words and linguistic syntagms. The meaning is represented by special real semisets. Because they are characterized by some indiscernibility relation, they can be technically approximated using fuzzy sets in the sense of [42]. Unfortunately, this book remained almost unnoticed for two reasons in different contexts: it was not mathematical enough and too non-standard for mathematicians, and it was too mathematical for linguists.

This book remained a permanent source of inspiration for the author in using formal fuzzy logic for various kinds of applications, especially when linguistic semantics is at play. The first direction pursued was that of *linguistically oriented control* [49], whose main idea was to develop a technique using which the computer could act as if it understood the expert's linguistic instructions regarding how a given plant should be controlled. The essential difference between this and (classical) fuzzy control is that the expert specifies his description of control directly using words *without thinking about fuzzy sets hidden inside*. This technique has been implemented and applied to control several real processes. The most successful was the control of an aluminium melting furnace in 1996 (see [52])[6].

Another direction was establishing the program of *fuzzy logic in broader sense* in [50]. The main idea was to develop a formal model of natural human reasoning with the emphasis on employing vague semantics of natural language and solving various related practical problems.

4.2 Entering of P. Hájek into fuzzy logic

For a long time, P. Hájek had been quite hesitant about fuzzy logic, arguing against it in many discussions with the author of this paper over the course of about 10 years. During one of these discussions in 1990 at the Conference of the Czech Mathematicians and Physicists in Zvíkovské Podhradı, he let himself be persuaded that fuzzy logic is not probability theory in disguise. Another point in favor of fuzzy logic came from G. Takeuti, namely from his paper [71] and from his visit to Prague.

After that time, P. Hájek's name was seen more and more often in connection with fuzzy logic (see, e.g. [18, 19, 30]. In the paper [20], he for the first time clearly formulated his view of fuzzy logic as a special branch of mathematical logic.

The event of crucial importance for the further development of mathematical fuzzy logic was the monograph [27]. In it, P. Hájek established mathematical fuzzy logic as a generalization of classical logic. This logic has slightly modified syntax (through the addition of new connectives and different set of axioms) and non-classical semantics. He actually introduced a class of formal logical systems now called (mathematical)

[6] It should be noted that this application turned out to be very successful and was extended to four more furnaces in subsequent years. The application has already been continuously working for more than 13 years.

fuzzy logic in narrow sense (FLn) with traditional syntax. In addition to G. Takeuti, he was inspired mainly by the papers of J. Pavelka, V. Novák, U. Höhle [37, 38], and D. Mundici on MV-algebras (e.g. [41]), as well as S. Gottwald's book on many-valued logic [17], and also by the people developing the mathematical theory of t-norms, namely R. Mesiar, E. P. Klement, E. Pap, D. Butnariu, M. Navara, and others (an essential monograph about t-norms that was published at the turn of the 21st century is [40]).

Fuzzy logic in P. Hájek's terms stems from the assumption that the truth values form a special residuated lattice that then yields rules for the specification of its syntax and semantics. Furthermore, the attempt to coin the interval of reals $[0,1]$ as a set of truth values led him to establish his *basic fuzzy logic* (BL-fuzzy logic, or simply BL). Based on the Moster-Shields theorem for t-norms stating that each continuous t-norm is either isomorphic to minimum, product, or Łukasiewicz conjunction (which is also a t-norm) or otherwise an ordinal sum of them (see [40] for details), he could confine to four kinds of main fuzzy logic: *Gödel logic*, which was initiated by K. Gödel and further developed by G. Takeuti in [71]; *Łukasiewicz logic*, which has been studied already for many years; *product logic*, which was established as a new logic by P. Hájek; and BL-fuzzy logic, comprised of all three previous ones. He also recognized the essential role of the prelinearity axiom

$$(a \to b) \vee (b \to a) = \mathbf{1}$$

which then enabled him to prove the following *completeness theorem* for all of the above-discussed logics: For every formula, A the following is equivalent:

1. A is provable.
2. A is a $\mathbf{1}$-tautology w.r.t. the corresponding algebras of truth values.
3. A is a $\mathbf{1}$-tautology w.r.t. the corresponding *linearly ordered* algebras of truth values.

This theorem supports the idea that the truth values of fuzzy logics should always be linearly ordered. The famous *standard completeness theorem* stating that semantics of BL-fuzzy logic can be confined only to BL-algebras on $[0,1]$ in item (iii) (cf. [8]) is a successful culmination of P. Hájek's ideas.

As for $\mathrm{Ev}_{Ł}$, P. Hájek never accepted the idea of evaluated syntax; thus, this logic was not for him a "genuine" logic. He reformulated it inside Łukasiewicz logic as a side logic among other fuzzy ones and calls it *Rational Pavelka Logic*. In a more or- ess artificial way, it is possible to introduce a special degree called the *provability degree*. The author of this paper, however, does not think that this term is convenient because it occludes the essential idea that axioms need not be fully convincing and thus that evaluated syntax is obtained as a nice generalization of the classical one. Namely, it can be demonstrated (cf. [53]) that the concept of provability degree can be seen as a natural generalization of classical provability and, consequently, that traditional

syntax is a special case of the evaluated one. From this point of view, J. Pavelka left the realm of classical logic in a most radical way, thus introducing a new kind of logic. On the other hand, there is only one $\mathrm{Ev}_{Ł}$ (cf. our discussion on $\mathrm{Ev}_{Ł}$ above), thus it seems to be too narrow for many mathematicians.

4.3 Ending of the decade

P. Hájek's monograph is the first of a series of books providing, in our opinion, a full picture of the state of the art in fuzzy logic in the nineties. The second book is [53], written by V. Novák, I. Perfilieva and J. Močkoř. It contains a complete presentation of $\mathrm{Ev}_{Ł}$. In addition, the book also contains detailed elaboration of the approximation properties of fuzzy models, the model theory of fuzzy logics in categories, and a more detailed elaboration of FLb (at that time, developed on the basis of $\mathrm{Ev}_{Ł}$ only). A notable result by I. Perfilieva is an original constructive proof of the McNaughton theorem for Łukasiewicz logic, which states that each formula of the latter can be uniquely represented by a piecewise linear function on $[0, 1]$.

The third book of this series is the volume [54] edited by V. Novák and I. Perfilieva. It contains contributions by several authors[7] and is focused on various further aspects of fuzzy logic not covered by the previous two books. A full picture of the state of the art of mathematical fuzzy logic in 20^{th} century is completed by the book [40] on t-norms written by E. P. Klement, R. Mesiar and E. Pap.

The successful development of fuzzy logic in the Czech Republic also opened the door to an institutional establishment. Namely, in 1996, the Institute for Research and Applications of Fuzzy Modeling was established in the University of Ostrava in Ostrava (IRAFM OU). Its founders were V. Novák, J. Močkoř, A. Dvořák, R. Bělohlávek, R. Smolíková and D. Jedelský. It became a renowned place of work that successfully combines theoretical and applied research in one place.

In 1999, IRAFM OU was significantly strengthened by Russian mathematician I. Perfilieva, who naturalized in the Czech Republic. She introduced many new impulses and, in addition to her work in fuzzy logic, she introduced an interesting and related area: the *fuzzy approximation* of functions.

At the turn of the century, a group of young people gathered around P. Hájek, mostly at the Institute of Computer Science of the Academy of Sciences of the Czech Republic in Prague (ICS AV ČR). Among them, very active people include P. Cintula and L. Běhounek, R. Horčík, Z. Haniková, D. Coufal and T. Kroupa (he works at the Inst. of Information Theory and Automation of the ASCR).

Both places of work became leading centers of research in fuzzy logic, not only in the Czech Republic, but also internationally Their researchers meet regularly several times per year and organize various meetings and conferences. The most important

[7] The contributiors were: L. A. Zadeh, G. J. Klir, E. P. Klement, R. Mesiar, E. Pap, J. Paris, A. Di Nola, G. Georgescu, I. Leuştean, E. Trillas, A. R. de Soto, S. Cubillo, O. Hájek, G. Gerla, S. Lehmke, E. Turunen, V. Novák, I. Perfilieva, H. Thiele, D. Mundici, F. Esteva, P. Garcia, L. Godo, S. Gottwald, F. Klawonn, R. Bělohlávek, K. Broda, A. Russo, D. Gabbay

event in the nineties was the Seventh IFSA'97 (International Fuzzy Systems Association) World Congress, which took place in June 25-29, 1997 and was co-organized by the University of Ostrava, the School of Economics in Prague, the Institute of Computer Science ASCR, the Institute of Theory of Information and Automation ASCR in Prague, Czech the Technical University in Prague, Slovak Technical University in Bratislava, and Silesian University in Karviná. The general chair was V. Novák, and the other organizers were J. Ivánek, M. Mareš, R. Mesiar, P. Hájek, J. Ramík, P. Vysoký, P. Přibyl, and M. Zeithamlová. The IFSA'97 World Congress in Prague was evaluated by many of its participants as the best of all IFSA congresses. The proceedings was published by the famous publishing house Academia in Prague [51].

5 21st Century

5.1 P. Hájek and his group in fuzzy logic

Thanks to P. Hájek's monograph (and also to the other mentioned books), the beginning of 21st was devoted to the intensive study of mathematical fuzzy logic not only in the Czech Republic but also in several other European countries, especially Italy and Spain. The results penetrated to a great depth both in the corresponding algebraic structures of truth values and in the properties of the corresponding formal logical systems. Various other logics have been studied, for example, ΠMTL by R. Horčík in [35, 36]. The work led to the combination of various axioms and was concluded in the so-called ŁΠ fuzzy logic initiated by Spanish and Italian colleagues in [16]. Their logic has been extended to a predicate first-order version and elaborated on in detail by P. Cintula in [9, 10].

One of the problems intensively studied by P. Hájek since the 1990s is complexity of fuzzy logics. This study was begun in [21] and then continued in a sequence of papers [22–26]. The most recent one is [12]. The complexity properties of fuzzy logic are very intricate and depend on the chosen class of logics and their semantics. The following sets of formulas have been studied: 1-tautologies, 1-satisfiable formulas (i.e., those that are 1-true for at least one model), positive tautologies (true in a degree > 0) and positively satisfiable formulas. Their complexity ranges, in general, from Σ_1 to Π_2, but in some cases, it is even non-arithmetical (usually if the given logic contains the product connective: e.g., BL). For example, Łukasiewicz predicate logic with respect to the standard algebra (2) has Π_2-complete set of 1-tautologies, Π_1-complete set of 1-satisfiable formulas, Σ_1-complete set of positive tautologies and Σ_1 complete set of positively satisfiable formulas. If we consider general tautologies (i.e., tautologies with respect to linearly ordered algebras from the given variety) then the given fuzzy logic is only Σ_1-complete. Moreover, for each set $\mathcal{K}$ of continuous t-norms such that propositional $\mathcal{K}$-tautologies are axiomatizable, general tautologies with respect to $\mathcal{K}$ are Σ_1-complete (cf. [25]). It is not surprising that complexity of fuzzy logics is sometimes fairly high given that fuzzy logics should be used in modeling various complex phenomena, namely if vagueness is at play and one cannot expect miracles. Thus, to apply it in practice, we must seek special local solutions.

Various kinds of fuzzy logics and their structure have been extensively studied. The most comprehensive study is the paper [13] written by P. Hájek, P. Cintula and R. Horčík, who studied 60 various systems of fuzzy logic simultaneously. A similar comprehensive paper on algebraic semantics of fuzzy logic is [14].

The fact that there are so many systems of fuzzy logic has inspired discussion about how fuzzy logic can be defined. In [11], P. Cintula proposes to delineate fuzzy logic mathematically, namely as a logic of the comparative notion of truth. A special issue entitled "What is Fuzzy Logic" from the journal Fuzzy Sets and Systems (Volume 157, Issue 5, Pages 595-718, March 2006) also contains the opinion of P. Hájek's group in the papers [5,28]. This discussion led them to the concept of *core fuzzy logic* (see [29]): it is a logic that is an expansion of MTL logic.fulfiling

$$A \Leftrightarrow B \vdash C(A) \Leftrightarrow C(B),$$

(for all formulas A, B, C) and holding the *local deduction theorem* (cf. (4)):

$$T \cup \{A\} \vdash B \quad \text{iff} \quad \text{there is } n > 0 \text{ such that } T \vdash A^n \Rightarrow B$$

(for all formulas A, B). One of the problems coming out of a formal definition is the following: one can argue that not all logics fulfilling it are indeed capable of providing a model of the vagueness phenomenon (e.g., the Gödel logic). This leads to an informal, taxative definition —fuzzy logic is one of several explicitly named logics chosen from the point of view of their capability to provide a model of essential features of the vagueness phenomenon and the human way of reasoning. A contribution to this direction is presented in [55,59].

A special direction that is very important for the development of fuzzy set theory is the program of *formal fuzzy mathematics* introduced by P. Cintula and L. Běhounek in [4]. Their idea is to develop various areas of fuzzy mathematics formally using mathematical fuzzy logic in the style introduced by P. Hájek as the metamathematical frame. The mathematical background, which is based on the class of fuzzy logics developed before, is the *fuzzy class theory* [3]. In a sense, this is a somewhat restricted higher-order fuzzy logic that might be equivalent with the fuzzy type theory mentioned below. Recall that the program of formal fuzzy mathematics is a precise formulation of the program anticipated already in the book [44].

P. Cintula and L. Běhounek not only introduced the program of fuzzy mathematics but also established a group of people (T. Kroupa, R. Horčík, M. Daňková, C. Noguera and few other ones) who began to realize it step by step. In any case, this is a significant attempt to put fuzzy set theory on very well and strongly established grounds.

5.2 Other activities in fuzzy logic in the Czech Republic

Several PhD theses have been defended dealing with various aspects of fuzzy logic, considerig the fundamental issues (P. Cintula), algebraic aspects (R. Horčík, Z. Haniková),

generalized quantifiers (M. Holčapek), normal forms of approximation theory (M. Daňková) and applications (A. Dvořák, M. Štěpnička).

The work of IRAFM OU in fuzzy logic has focused on several aspects. In 2005, higher-order fuzzy logic was introduced by V. Novák in [56]. In accordance with the classical type theory[8] it is called the *fuzzy type theory*. In principle, the λ-calculus has been extended and many-valued semantics has been introduced for it. The main motivation was to introduce a sound formal means for the development of FLb, which would be more convenient than $\mathrm{Ev}_{Ł}$ as applied so far. Let us remark that everything formulated in formal fuzzy mathematics using the fuzzy class theory can also easily be formulated in the fuzzy type theory (using the λ-calculus).

Among the topics which were elaborated more deeply, let us mention the concept of *fuzzy logic normal forms* in predicate fuzzy logic and their use in the fuzzy approximation: i.e., a formal theory of approximation of functions on the basis of imprecise information. This concept was initiated by I. Perfilieva already in [65] and elaborated on further by M. Daňková (cf. [15, 64, 66, 67]).

In 2006, the Russian edition of the book by V. Novák, I. Perfilieva and J. Močkoř was published in [58]. The book has been quite successful in this vast market. Notable results have been obtained also in the theory of generalized (fuzzy) quantifiers by M. Holčapek and A. Dvořák [33, 34], who introduced the results of the "classical theory" of generalized quantifiers (cf. [68]) into mathematical fuzzy logic, and V. Novák [57], who proposed a formal model of so-called intermediate quantifiers (e.g., "most, many, a lot of, several, a few", etc.).

Let us also mention the attempt at leaving the realm of residuated lattice-based fuzzy logic by introducing a new special algebraic structure called EQ-algebra, in which the fundamental operation is that of *fuzzy equality* $\sim$. The implication in it is a derived operation (cf. [60–62]) defined by

$$a \to b = (a \wedge b) \sim a. \tag{5}$$

This direction in some sense follows the ideas of G. W. Lebniz, who claimed that "a fully satisfactory logical calculus must be an equational one". As a matter of fact, formula (5) also follows the Leibnitz idea[9].

Another small group led by R. Bělohlávek (he left IRAFM OU in 2004) was established in Olomouc. Let us refer especially to two books [2] and [6] where the fuzzy equational logic has been established. It has been shown in this book that when confining only to equalities, generalized completeness in the sense of $\mathrm{Ev}_{Ł}$ can be obtained even without the assumption that the implication operation is continuous on $[0,1]$.

Several scientific meetings have taken place in the Czech Republic: in 2005, the conference "The Logic of Soft Computing V" was organized in Ostrava by IRAFM

[8] The type theory was introduced already by B. Russel and A. Whitehead in their Principia Mathematica; it was later on developed more formally by A. Church in [7] and elaborated in detail by L. Henkin in [31,32] and P. Andrews in [1].

[9] Let $A \equiv B$ be an equality of concepts and AB their intersection. Then Leibnitz interprets that "if A then B" as $AB \equiv A$.

OU. The conference was the penultimate of the series of this kind of conferences on formal fuzzy logic[10]. A special issue of some selected papers has been published in Fuzzy Sets and Systems vol. 158, 2007.

An attempt at organizing a meeting of philosophers and mathematicians in one place was "Conference on Uncertainty & Vagueness" which took place in Prague, September 5-8, 2006. The main goal of this conference was to make philosophers familiar with mathematical fuzzy logic and to explain to them that their criticism of the latter is unjustified (cf. [39]). Unfortunately, it seems that the philosophers (though hopefully not all of them) are to a great extent closed towards ideas coming from outside and thus, from this point of view, the conference was not fully successful.

The most important meeting in the first decade of the 21st century was the 5^{th} EUSFLAT Conference (European Society for Fuzzy Logic and Technology), which took place in Ostrava (Czech Republic) during September 11-14, 2007. The organizer of this conference was IRAFM OU. The conference was successful and the proceedings were published in [70]. In this conference, the Working Group on Mathematical Fuzzy Logic (MathFuzzLog) of EUSFLAT was established. It aims to conduct and promote research in Mathematical Fuzzy Logic.

It is difficult to write a complete presentation of the results related fuzzy logic in the Czech Republic in 21st century because many activities of various kinds have been accomplished and so, much more space than we have would be needed. We refer the readers to the webpages http:\\irafm.osu.cz\ and http:\\www.cs.cas.cz\mathfuzzlog\ where more information, including full references to many publications or to their preprints, can be found.

6 Conclusion

This paper aims especially to pay tribute to P. Hájek and his work in fuzzy logic on the occasion of 70th birthday celebration. We have briefly overviewed some of the main points in the development of fuzzy logic in Czechoslovakia and, since 1993, in the Czech Republic. Our presentation is far from being complete because we focused mainly on fuzzy logic and because we omitted a lot of results obtained in other areas of fuzzy mathematics such as fuzzy mathematical programming and other ones.

Our goal was to show that the contribution of P. Hájek to fuzzy logic is crucial. Without him, the subject would be incomparably less developed. He deserves many thanks for his work. We wish him good health and many inspiring ideas in the following years.

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[10]The first one was organized in Capri in 2001 and the last one in Malaga in 2006.

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